Real Analysis (Folland) Exercises

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1 Chapter 1: Measures

Exercise 2 Show that $\mathcal{B}_{\mathbb{R}}$ is generated by each of the following:
(a) the open intervals $\mathcal{E}_1 = \{(a,b) : a < b\}$,
(b) the closed intervals $\mathcal{E}_2 = \{[a,b] : a < b\}$,
(c) the half-open intervals $\mathcal{E}_3 = \{(a,b] : a < b)\}$ or $\epsilon_3 = \{(a,b] : a < b)\}$,
(d) the open rays $\mathcal{E}_5 = \{(a,\infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty,a) : a \in \mathbb{R}\}$,
(e) the closed rays $\mathcal{E}_7 = \{[a,\infty)\ a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty,a] : a \in \mathbb{R}\}$.

Proof. Recall lemma 1.1, which states if $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$. It is easy to observe $\mathcal{E}_i \subset \mathcal{B}_{\mathbb{R}}$, therefore $\mathcal{M}(\mathcal{E}_i) \subset \mathcal{B}_{\mathbb{R}}$.

(a) Since every open set can be written as a countable union of intervals, denote $\mathcal{O}$ as the set of all open sets, then $\mathcal{O} \subset \mathcal{M}(\mathcal{E}_1)$, $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_1)$. Hence $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1)$.
(b) Attempt to show $\mathcal{E}_1 \subset \mathcal{M}(\mathcal{E}_2)$. Apparently $(a,b) = \cup_{n=1}^{\infty}[a+1/n, b-1/n]$.
(c) $\mathcal{E}_1 \subset \mathcal{M}(\mathcal{E}_3)$ since $(a,b) = \cup_{n=1}^{\infty}(a, b-1/n]$. The same goes for $\mathcal{E}_4$.
(d) $\mathcal{E}_3 \subset \mathcal{M}(\mathcal{E}_5)$ since $(a,b] = (a,\infty) \cap ((b,\infty))^c$. The same argument goes for $\mathcal{E}_6$.
(e) $\mathcal{E}_4 \subset \mathcal{M}(\mathcal{E}_7)$ since $[a,b) = [a,\infty] \cap ([b,\infty))^c$.
Therefore $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_i)$. □

Exercise 4 An algebra $\mathcal{A}$ is a σ-algebra iff $\mathcal{A}$ is closed under countable increasing unions.

Proof. If $\mathcal{A}$ is closed under countable increasing unions, for any countable collection of sets $\{F_j\}$ in $\mathcal{A}$, let $E_1 = F_1$, $E_2 = E_1 \cup F_2$, $E_n = E_{n-1} \cup F_n$, then $\{E_j\}$ is an increasing sequence of sets, therefore $\cup_{n=1}^{\infty} E_n = \cup_{j=1}^{\infty} F_j \in \mathcal{A}$. Therefore $\mathcal{A}$ is a σ-algebra. The reverse is trivial. □

Exercise 5 If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.

Proof. Let A be the index set of all countable subsets of $\mathcal{E}$. First claim that $\mathcal{B} = \cup_{\alpha \in A}\mathcal{M}(\mathcal{F}_\alpha)$ is a σ-algebra. $\forall\, E \in \mathcal{B}$, $E \in \mathcal{M}(\mathcal{F}_\alpha)$, therefore $E^c \in \mathcal{B}$. Given a countable collection of sets $\{E_j\}$ in $\mathcal{B}$, since $E_j \in \mathcal{M}(F_\alpha)$, E_j must be in at least one $\mathcal{M}(\mathcal{F}_j)$. Let $\mathcal{H} = \cup_{j=1}^{\infty}\mathcal{F}_j$, consider $\mathcal{M}(\mathcal{H})$. Obviously $\{E_j\} \in \mathcal{M}(\mathcal{H})$, therefore $\cup_{j=1}^{\infty} E_j \in \mathcal{M}(\mathcal{H})$. Since $\mathcal{H}$ is also a countable subset of $\mathcal{E}$, $\mathcal{M}(\mathcal{H}) \subset \mathcal{B}$. Therefore $\mathcal{B}$ is indeed a σ-algebra.

It is straightforward that $\mathcal{E} \subset \mathcal{B}$. For the reverse, $\forall\, E \in \mathcal{B}$, E is in some σ-algebra generated by $\mathcal{F}_\alpha$, therefore $E \in \mathcal{M}$. Thus $\mathcal{M} = \mathcal{B}$. □

Exercise 6 Suppose that (X, M, μ) is a measure space. Let $\mathcal{N} = \{N \in M : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M}, F \subset N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

Proof. Apparently $\overline{\mathcal{M}}$ is closed under countable unions. For any $E \in \mathcal{M}, F \subset N \in \mathcal{N}$, without the loss of generality assume $E \cap N = \varnothing$ (otherwise replace N, F with $N \backslash E$ and $F \backslash E$). Then $E \cup F = (E \cup N) \cap (N^c \cup F)$, $(E \cup F)^c = (E \cup N)^c \cup (N \cap F^c)^c \in \overline{\mathcal{M}}$. Therefore $\overline{\mathcal{M}}$ is a σ-algebra.

Now consider the extension $\overline{\mu}$. Let $\overline{\mu}(E \cup F) = \mu(E)$. This is well-defined since if $E_1 \cup F_1 = E_2 \cup F_2$ then $E_1 \subset E_2 \cup N_2$, $\mu(E_1) \leq \mu(E_2)$, and likewise $\mu(E_1) \geq \mu(E_2)$, thus $\mu(E_1) = \mu(E_2)$. Then $\overline{\mu}(\varnothing) = \overline{\mu}(\varnothing \cup \varnothing) = 0$, and the countable additivity can be likewise easily verified. For the uniqueness, give any other measure $\overline{\mu}'$, $\overline{\mu}'(E \cup F) \leq \overline{\mu}'(E \cup N) \leq \mu(E)$. But $\overline{\mu}'(E \cup F) \geq \overline{\mu}'(E \cup \varnothing) = \mu(E)$, thus $\overline{\mu}' = \overline{\mu}$. □

Exercise 7 If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, \ldots, a_n \in [0, \infty)$, then $\sum_1^n a_j \mu_j$ is a measure on $(X, \mathcal{M})$.

Proof. Let $\mu' = \sum_1^n a_j \mu_j$. Then $\mu'(\varnothing) = 0$, given any collection disjoint sets $\{E_j\}$ in $\mathcal{M}$, $\mu'(\cup_1^\infty E_j) = \sum_1^n a_j \mu_j(\cup_1^\infty E_j) = \sum_{j=1}^\infty \sum_1^n a_j \mu_j(E_j) = \sum_{j=1}^\infty \mu'(E_j)$, therefore μ' is also a measure. □

Exercise 8 If $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided that $\mu(\cup_1^\infty E_j) < \infty$.

Proof. Recall

$$\liminf E_j = \bigcup_{j=1}^\infty \bigcap_{i=j}^\infty E_i, \quad \limsup E_j = \bigcap_{j=1}^\infty \bigcup_{i=j}^\infty E_i$$

observe that $\{A_j = \cap_{i=j}^\infty E_j\}$ gives a sequence such that $A_1 \subset A_2 \cdots$, since μ is a measure, $\mu(\liminf E_j) = \mu(\cup_{j=1}^\infty A_j) = \lim_{j\to\infty} \mu(A_j) \leq \liminf \mu(E_j)$. For the second claim, in the same sense let $\{B_j = \cup_{i=j}^\infty E_j\}$, then $\mu(\limsup E_j) = \lim_{j\to\infty} \mu(B_j) \geq \limsup \mu(B_j)$. □

Exercise 9 If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

Proof. Since μ is a measure, $\mu(E)+\mu(F) = \mu(E\cap F)+\mu(E\cap F^c)+\mu(E\cap F)+\mu(E^c\cap F) = \mu(E\cup F)+\mu(E\cap F)$. □

Exercise 10 Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$. Then μ_E is also a measure.

Proof. Apparently $\mu_E(\varnothing) = 0$. Given any collection of disjoint sets $\{A_j\}$ in $\mathcal{M}$, $\mu_E(\cup_{j=1}^\infty A_j) = \mu(\cup_{j=1}^\infty A_j \cap E) = \mu(\cup_{j=1}^\infty (A_j \cap E)) = \sum_{j=1}^\infty \mu_E(A_j)$. Therefore μ_E is a measure. □

Exercise 11 A finitely additive measure μ is a measure iff it is continuous from below. If $\mu(X) < \infty$, μ is a measure iff it is countinuous from above.

Proof. Given a finitely additive measure μ, if it is continuous from below, then given a sequence of disjoint sets $\{E_j\}$, let $\{A_j = \cup_{i=1}^j E_i\}$, $\mu(\cup_j E_j) = \mu(\cup_j A_j) = \lim_{n\to\infty} \mu(A_n)$, by finite additivity $\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \sum_{i=1}^n \mu(E_i) = \sum_{n=1}^\infty \mu(E_j)$. For the second claim, let $\{B_j = A_j^c\}$, then $\cup_j E_j = \cup_j A_j = (\cap_j (A_j^c))^c = (\cap_j (B_j))^c$, therefore $\mu(\cup_j E_j) + \mu(\cap_j B_j) = \mu(X)$. By continuity from above, $\mu(\cap_j B_j) = \lim_{j\to\infty} \mu(B_j) = \mu(X) - \lim_{j\to\infty} \mu(A_j)$, the rest is the same as the previous argument. □

Exercise 12 Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

(a) If $E, F \in \mathcal{M}$ and $\mu(E \triangle F) = 0$, then $\mu(E) = \mu(F)$.

(b) Say that $E \sim F$ if $\mu(E \triangle F)$. Then $\sim$ is an equivalence relation on $\mathcal{M}$.

(c) For $E, F \in M$, define $\rho(E, F) = \mu(E \triangle F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space $M/\sim$.

Proof. Recall $E \triangle F = (E \backslash F) \cup (F \backslash E)$.

(a) Since $E \backslash F$ and $F \backslash E$ are disjoint, $\mu(E \triangle F) = \mu(E \backslash F) + \mu(F \backslash E)$. Since $E = (E \backslash F) \cup (E \cap F)$, $F = (F \backslash E) \cup (E \cap F)$, $\mu(E \triangle F) = \mu(E) + \mu(F) - 2\mu(E \cap F)$. Notice that $\mu(E) \geq \mu(E \cap F)$, $\mu(F) \geq \mu(E \cap F)$, therefore when $\mu(E \triangle F) = 0$, $\mu(E) = \mu(F) = \mu(E \cap F)$.

(b) Since "=" is an equivalence relation on $[0, \infty)$, "$\sim$" is obviously also an equivalence relation.

(c) Attempt to verify $\mu(E \triangle G) \leq \mu(E \triangle F) + \mu(F \triangle G)$:

$$\begin{aligned}
\mu(E \triangle G) &= \mu(E \backslash G) + \mu(G \backslash E) \\
&= \mu(E) + \mu(G) - 2\mu(E \cap G) \\
&\leq \mu(E) + 2\mu(F) + \mu(G) - 2\mu(E \cap F) - 2\mu(F \cap G) \\
&= \mu(E \triangle F) + \mu(F \triangle G)
\end{aligned}$$

where the inequality $\mu(F) + \mu(E \cap G) = \mu(F \cap E^c \cap G^c) + \mu(F \cap G^c \cap E) + \mu(F \cap G \cap E^c) + 2\mu(F \cap G \cap E) + \mu(E \cap G \cap F^c) \geq \mu(F \cap E \cap G^c) + \mu(F \cap G \cap E^c) + 2\mu(E \cap F \cap G) = \mu(E \cap F) + \mu(F \cap G)$ is utilized. $\square$

Exercise 14 If μ is a semifinite measure and $\mu(E) = \infty$, for any $C > 0$ there exists $F \subset E$ with $C < \mu(F) < \infty$.

Proof. Assume that there exists $C > 0$ such that $\forall F \subset E$, $\mu(F) \leq C$, then $\sup\{\mu(F) : F \subset E\} \leq C$. Denote the supremum with S. Then $\forall n \in \mathbb{N}, \exists F_n \subset E$ such that $S - 1/n < \mu(F_n) \leq S$. Since $F' = \cup_{n=1}^{\infty} F_n \subset E$, $\mu(F') = S$. Then consider $E \backslash F'$. Obviously $\mu(E \backslash F') = \infty$. Because μ is semifinite, there exist F'' such that $0 < \mu(F'') < \infty$. Then $\mu(F' \cup F'') > S$, contradiction. Therefore there is no supremum. $\square$

Exercise 15 Given a measure μ on $(X, \mathcal{M})$, define μ_0 on $\mathcal{M}$ by $\mu_0(E) = \sup\{\mu(F) : F \subset E \text{ and } \mu(F) < \infty\}$.

(a) μ_0 is a semifinite measure. It is called the semifinite part of μ.

(b) If μ is semifinite, then $\mu = \mu_0$.

(c) There is a measure ν on $\mathcal{M}$ (in general, not unique) which assumes only values 0 and ∞ such that $\mu = \mu_0 + \nu$.

Proof. (a) First verify that μ_0 is a measure. Obviously $\mu_0(\varnothing) = 0$. Give any collection of disjoint sets $\{E_j\}$, let $E = \cup_{j=1}^{\infty} E_j$. For a measurable set $F \subset E$ and $\mu(F) < \infty$, $\mu(F) = \sum_j \mu(F \cap E_j) \leq \sum_j \mu_0(E_j)$. Since this holds for any subset of E that has finite measure, $\mu_0(E) \leq \sum_j \mu_0(E_j)$. If $\mu_0(E) = \infty$, then the reverse trivially holds. Otherwise $\mu_0(E) < \infty$. Then for each E_j, $\forall \epsilon/2^j$, there exists $F_j \subset E_j$ such that $\mu_0(E_j) - \epsilon/2^j < \mu(F_j) \leq \mu_0(E_j)$. Then $\mu_0(E) \geq \mu(\cup_{j=1}^{\infty} F_j) = \sum_j \mu_0(E_j) - \epsilon$. Therefore $\mu_0(E) = \sum_j \mu_0(E_j)$, μ_0 is a measure.

Given a E such that $\mu_0(E) = \infty$, take any $C > 0$, then $\exists F \subset E$ such that $C < \mu(F) < \infty$. Then $\mu_0(F) = \mu(F)$ is non-zero and finite. Therefore μ_0 is a semifinite measure.

(b) For any $E \in \mathcal{M}$, if $\mu(E) < \infty$, then $\mu(E) = \mu_0(E)$. If $\mu(E) = \infty$, then by Exercise 14 $\mu_0(E) = \infty$. Therefore $\mu = \mu_0$.

(c) Let

$$\nu(E) = \begin{cases} 0, & \text{if } E \text{ is } \sigma\text{-finite} \\ \infty, & \text{otherwise} \end{cases}$$

ν is a measure since the disjoint union of σ-finite sets is still a σ-finite set, and if there is a set that is not σ-finite in the collection the union will also not be σ-finite. Now verify $\mu(E) = \mu_0(E) + \nu(E)$. When E is σ-finite, if $\mu(E)$ is finite, then the quality holds. If $\mu(E)$ is not finite, then by previous exercise $\mu_0(E) = \infty$, the quality still holds. If E is not σ-finite, the quality holds trivially. $\square$

Exercise 16 Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subset X$ is called locally measurable if for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$, $E \cap A \in \mathcal{M}$. Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable sets. Clearly $\mathcal{M} \subset \widetilde{\mathcal{M}}$; if $\mathcal{M} = \widetilde{\mathcal{M}}$, then μ is called saturated.

(a) If μ is σ-finite, then μ is saturated.

(b) $\widetilde{\mathcal{M}}$ is a σ-algebra.

(c) Define $\widetilde{\mu}$ on $\widetilde{\mathcal{M}}$ by $\widetilde{\mu} = \mu(E)$ if $E \in \mathcal{M}$ and $\widetilde{\mu}(E) = \infty$ otherwise. Then $\widetilde{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$, called the saturation of μ.

(d) If μ is complete, so is $\overline{\mu}$.

(e) Suppose that μ is semifinite. For $E \in \widetilde{\mathcal{M}}$, define $\underline{\mu}(E) = \sup\{\mu(A) : A \in \mathcal{M} \text{ and } A \subset E\}$. Then $\underline{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ that extends μ.

(f) Let X_1, X_2 be disjoint uncountable sets, $X = X_1 \cup X_2$, and $\mathcal{M}$ the σ-algebra of countable or co-countable sets in X. Let μ_0 be counting measure on $\mathcal{P}(X_1)$ and define μ on $\mathcal{M}$ by $\mu(E) = \mu_0(E \cap X_1)$. Then μ is a measure on $\mathcal{M}$, $\widetilde{M} = \mathcal{P}(X)$, and in the notation of (c) and (e), $\widetilde{\mu} \neq \underline{\mu}$.

Proof. (a) Since μ is σ-finite, there exists a countable collection of disjoint sets $\{E_j\}$ such that $X = \cup_{j=1}^{\infty} E_j$ and $\mu(E_j) \leq \infty$. Therefore $\forall\, E \in \widetilde{\mathcal{M}}$, for each E_j, $E \cap E_j \in \mathcal{M}$. Thus $E = \cup_1^{\infty}(E \cap E_j) \in \mathcal{M}$. Hence $\widetilde{\mathcal{M}} = \mathcal{M}$.

(b) $\forall\, E \in \widetilde{\mathcal{M}}$, $\forall\, A \in \mathcal{M}$ such that $\mu(A) < \infty$, $E^c \cap A = (A \cap (E \cap A)^c) \in \mathcal{M}$, therefore $E^c \in \widetilde{\mathcal{M}}$. Give any countable collection of sets $\{E_j\}$ in $\widetilde{\mathcal{M}}$, for any $A \in \mathcal{M}$ that has finite measure, $(\cup_j E_j) \cap A = \cup_j (E_j \cap A) \in \mathcal{M}$. Thus $\widetilde{\mathcal{M}}$ is a σ-algebra.

(c) First check that $\widetilde{\mu}$ is a measure. Apparently $\widetilde{\mu}(\varnothing) = 0$. Given any countable collection of disjoint sets $\{E_j\}$ in $\widetilde{\mathcal{M}}$, if $E_j \in \mathcal{M}$ for each j, then the additivity trivially holds. If $\exists\, i$ such that $E_i \notin \mathcal{M}$, assume $\cup_j E_j \in \mathcal{M}$. Obviously $\cup_j E_j$ cannot have finite measure. Therefore the equality still holds. Then check that $\widetilde{\mu}$ is saturated. $\forall\, E$, if $\forall\, A \in \widetilde{\mathcal{M}}$ such that $\widetilde{\mu}(A) < \infty$, $A \cap E \in \widetilde{\mathcal{M}}$, then $\mu(A) < \infty$, therefore $\widetilde{\mu}(A \cap E) < \infty$, $A \cap E \in \mathcal{M}$, $E \in \widetilde{\mathcal{M}}$.

(d) $\forall\, N \in \widetilde{\mathcal{M}}$, if $\widetilde{\mu}(N) = 0$, then because μ is complete, $\forall\, F \subset N$, $\widetilde{\mu}(F) = 0$. Therefore $\widetilde{\mu}$ is also complete.

(e) First verify $\underline{\mu}$ is a measure. Obviously $\underline{\mu}(\varnothing) = 0$. Given any countable collection of disjoint sets $\{E_j\}$ in $\widetilde{\mathcal{M}}$, assume they are all finite. $\forall\, E_j$, $\exists\, A_j$ such that $A_j \in \mathcal{M}$, $A_j \subset E_j$, $\underline{\mu}(E_j) - \epsilon/2^j < \mu(A_j) \leq \underline{\mu}(E_j)$. Then $\underline{\mu}(\cup_j E_j) \geq \mu(\cup_j A_j) > \sum_j \underline{\mu}(E_j) - \epsilon$, therefore $\underline{\mu}(\cup_j E_j) \geq \sum_j \underline{\mu}(E_j)$. For the reverse inequality, take $A \in \mathcal{M}$ such that $\underline{\mu}(\cup_j E_j) - \epsilon < \mu(A) \leq \underline{\mu}(\cup_j E_j)$, since $A \subset \cup_j E_j$ and $\mu(A) < \infty$, $A_j = A \cap E_j \in \mathcal{M}$, therefore $\underline{\mu}(\cup_j E_j) - \epsilon < \mu(A) \leq \sum_j \underline{\mu}(E_j)$. Therefore the reverse inequality holds, $\underline{\mu}(\cup_j E_j) = \sum_j \underline{\mu}(E_j)$. For the infinite case, since μ is semifinite, by exercise 14 both inequality hold trivially. If $E \in \mathcal{M}$, then $\mu(E) = \underline{\mu}(E)$ since μ is semifinite. Therefore $\underline{\mu}$ is an extend of μ.

Now check that $\underline{\mu}$ is saturated. $\forall\, E \in \widetilde{\widetilde{\mathcal{M}}}$, $\forall\, A \in \mathcal{M}$ such that $\underline{\mu}(A) < \infty$, $E \cap A \in \widetilde{\mathcal{M}}$. Then $E \cap A = E \cap A \cap A \in \mathcal{M}$.

(f) Since μ_0 is a well-defined measure, it is straightforward that μ is also a measure. $\forall\, A \subset X$, given any B such that $B \in \mathcal{M}$ and $\mu(B) < \infty$, since $B \cap X_1$ is finite, B must be countable. Therefore $B \cap A$ is also countable, $B \cap A \subset \widetilde{\mathcal{M}}$. Therefore $\widetilde{\mathcal{M}} = \mathcal{P}(X)$. Obviously $\widetilde{\mu} \neq \underline{\mu}$, one example may be $\{x_1\} \cup X_2$ where $x_1 \in X_1$. □

Exercise 17 If μ^* is an outer measure on X and $\{A_j\}_1^{\infty}$ is a sequence of disjoint μ^*-measurable sets, then $\mu^*(E \cap (\cup_1^{\infty} A_j)) = \sum_1^{\infty} \mu^*(E \cap A_j)$ for any $E \subset X$.

Proof. Since $\cup_j (E \cap A_j) = E \cap (\cup_j A_j)$, $\mu^*(E \cap (\cup_1^{\infty} A_j)) \leq \sum_1^{\infty} \mu^*(E \cap A_j)$. For the reverse inequality, let $B_n = \cup_{i=1}^{n} A_i$. Then $\mu^*(E \cap B_n) = \mu^*(E \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) = \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) = \sum_{i=1}^{n} \mu^*(E \cap A_i)$. Since $\mu^*(E \cap B_\infty) \geq \mu^*(E \cap B_n) = \sum_1^n \mu^*(E \cap A_i)$ for any n, $\mu^*(E \cap (\cup_1^{\infty} A_j)) \geq \sum_1^{\infty} \mu^*(E \cap A_j)$. □

Exercise 18 Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

(a) For any $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.

(b) If $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \backslash E) = 0$.

(c) If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

Proof. (a) Recall the definition of the outer measure μ^* on X:

$$\mu^*(E) = \inf\{\sum_j \mu_0(A_j) : A_j \in \mathcal{A}, E \subset \cup_j A_j, j = 1, 2, \cdots\}$$

If $\mu^*(E) = \infty$, the inequality holds trivially. Consider the case where $\mu^*(E) < \infty$. Then $\forall\, \epsilon > 0$, $\exists\, \{A_j\}$ with $A_j \in \mathcal{A}$ for each j and $E \subset \cup_j A_j$ such that $\mu^*(\cup_j A_j) \leq \sum_j \mu^*(A_j) \leq \mu^*(E) + \epsilon$. Therefore take $A = \cup_j A_j$.

(b) If E is μ^*-measurable, then by the first claim given $\epsilon = 1/k$, $k \in \mathbb{N}$, there exists $A_k \in \mathcal{A}_\sigma$ such that $E \subset A_k$, $\mu^*(A_k) = \mu^*(A \cap E) + 1/k$. Let $B = \cap_k A_k$. It is obvious that $\mu^*(B) = \mu^*(E)$. Therefore $\mu^*(E) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c) = \mu^*(B)$, $\mu^*(B \backslash E) = 0$.

For the inverse, $\forall\, A \subset X$, $\mu^*(A \cap E) + \mu^*(A \cap E^c) \le \mu^*(A \cap B) + \mu^*(A \cap E^c \cap B^c) + \mu^*(A \cap E^c \cap B) \le \mu^*(A \cap B) + \mu^*(A \cap B^c)$. $\forall\, \epsilon > 0, \exists\, C \in \mathcal{A}_\sigma$ such that $\mu^*(C) \le \mu^*(A) + \epsilon$ and $A \subset C$. By Caratheodory's theorem μ^* is a measure on $\mathcal{M}(\mathcal{A})$, therefore $\mu^*(A) + \epsilon \ge \mu^*(C) = \mu^*(C \cap B) + \mu^*(C \cap B^c) \ge \mu^*(A \cap B) + \mu^*(A \cap B^c)$. Therefore $\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \cap E^c)$.

(c) Notice that only need to prove the forword direction given that $\mu^*(E) = \infty$. Since μ_0 is σ-finite, $\exists\, \{A_j\} \subset \mathcal{A}$ such that $\mu_0(A_j) < \infty$, $X = \cup_j A_j$. Let $E_j = E \cap A_j$, $\forall\, \epsilon > 0$, take $B_j \in \mathcal{A}_\sigma$ and $E_j \subset B_j$ such that $\mu^*(B_j) \le \mu^*(E_j) + \epsilon/2^j$, then $\mu^*(B \backslash E) \le \mu^*(\cup_j (B_j \backslash E_j)) \le \sum_j \mu^*(B_j \backslash E_j) = \sum_j (\mu^*(B_j) - \mu^*(E_j)) \le \epsilon$. Therefore $\mu^*(B \backslash E) = 0$. □

Exercise 19 Let μ^* be an outer measure on X induced from a finite premeasure μ_0. If $E \subset X$, define the inner measure of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$.

Proof. If E is μ^*-measurable, then $\mu_0(X) = \mu^*(X) = \mu^*(E) + \mu^*(E^c)$, hence $\mu^*(E) = \mu_*(E)$. For the inverse, given $\mu^*(E) + \mu^*(E^c) = \mu_0(X)$, by exercise 18, $\forall\, n \in \mathbb{N}, \exists\, A_n \in \mathcal{A}_\sigma$ such that $E \subset A_n$, $\mu^*(A_n) \le \mu^*(E) + 1/n$. Let $A = \cap_n A_n$, then $A \in \mathcal{A}_{\sigma\delta}$ with $E \subset A$. Since A_n is μ^*-measurable, $\mu^*(A \cap E^c) \le \mu^*(A_n \cap E^c) = \mu(E^c) - \mu(A_n^c \cap E) \le \mu_0(X) - \mu^*(E) - \mu(A_n^c) \le \mu^*(A_n) - \mu^*(E) \le 1/n$ for any n, thus $\mu^*(A \cap E^c) = 0$, therefore by exercise 18 E is μ^*-measurable. □

Exercise 20 Let μ^* be an outer measure on X, $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets, $\overline{\mu} = \mu^* | \mathcal{M}^*$, and μ^+ the outer measure induced by $\overline{\mu}$.

(a) If $E \subset X$, we have $\mu^*(E) \le \mu^+(E)$, with equality iff there exists $A \in \mathcal{M}^*$ with $E \subset A$ and $\mu^*(A) = \mu^*(E)$.

(b) If μ^* is induced from a premeasure, then $\mu^* = \mu^+$.

(c) If $X = \{0, 1\}$, there exists an outer measure μ^* on X such that $\mu^* \ne \mu^+$.

Proof. (a) By the construction of the outer measure, if $\mu^+(E) < \infty$, then $\forall\, \epsilon > 0, \exists\, E_j$ with $E_j \in \mathcal{M}^*$ for each j, and $E \subset \cup_j E_j$ such that $\mu^*(E) \le \sum_j \mu^*(E_j) \le \mu^+(E) + \epsilon$, therefore $\mu^*(E) \le \mu^+(E)$. For the second claim, when $\mu^*(E) = \mu^+(E)$, one may take $E_j \in \mathcal{M}^*$ such that $\{E_j\}$ covers E and $\mu^*(E) = \mu^+(E) = \sum_j \mu^*(E_j)$. Thus just take $A = \cup_j E_j$. For the reverse, since A covers E, $\mu^*(E) \le \mu^+(E) \le \mu^*(A)$. By $\mu^*(E) = \mu^*(A)$ the equality must be taken.

(b) Since μ^* is induced from a premeasure, by exercise 18, for any $n \in \mathbb{N}$, there exists $A_n \in \mathcal{M}^*$ such that $E \subset A_n$ and $\mu^*(E) \le \mu^*(A_n) \le \mu^*(E) + 1/n$. Let $A = \cap_n A_n$, then $A \in \mathcal{M}^*$ with $E \subset A$ and $\mu^*(A) = \mu^*(E)$. By (a) $\mu^*(E) = \mu^+(E)$ for any $E \subset X$.

(c) Since $\mathcal{P}(X) = \{\varnothing, \{0\}, \{1\}, \{0, 1\}\}$, and $\mu^*(\varnothing) = \mu^+(\varnothing) = 0$, let

$$\mu^*(\{0\}) = a, \quad \mu^*(\{1\}) = b, \quad \mu^*(\{0, 1\}) = c$$

because of monotonicity, $0 \le a \le c$, $0 \le b \le c$. Then by subadditivity, $a + b \ge c$. If $\{0\}$ or $\{1\}$ is μ^*-measurable, then $\mathcal{M}^* = \mathcal{P}(X)$, $\overline{\mu} = \mu^* = \mu^+$. Therefore they must not be μ^*-measurable, $a + b \ne c$. Then $\mu^+(\{0\}) = \mu^+(\{1\}) = c$, $\mu^* \ne \mu^+$. □

Exercise 21 Let μ^* be an outer measure induced from a premeasure and $\overline{\mu}$ the restriction of μ^* to the μ^*-measurable sets. Then $\overline{\mu}$ is saturated.

Proof. Give a set $E \subset X$ such that $\forall\, A$ that is μ^*-measurable, $E \cap A$ is still μ^*-measurable and $\mu^*(A) < \infty$, now show that E is μ^*-measurable. For any $F \subset X$ that $\mu^*(F) < \infty$, $\exists\, \epsilon > 0$ such that $A \in \mathcal{A}_\sigma$ such that $F \subset A$ and

$$\begin{aligned}
\mu^*(F) + \epsilon \ge \mu^*(A) &= \mu^*(A \cap (A \cap E)) + \mu^*(A \cap (A \cap E)^c) \\
&= \mu^*(A \cap E) + \mu^*(A \cap E^c) \ge \mu^*(F \cap E) + \mu^*(F \cap E^c)
\end{aligned}$$

therefore E is μ^*-measurable. □

Exercise 22 Let $(X, \mathcal{M}, \mu)$ be a measure space, μ^* the outer measure induced by μ, $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets, and $\overline{\mu} = \mu^*|\mathcal{M}^*$

(a) If μ is σ-finite, then $\overline{\mu}$ is the completion of μ.

(b) In general, $\overline{\mu}$ is the saturation of the completion of μ.

Proof. (a) Since μ is σ-finite, if $E \in \mathcal{M}^*$ then $\exists\, B \in \mathcal{M}$ such that $E \subset B$ and $\mu^*(B \backslash E) = 0$. Therefore for any $n \in \mathbb{N}$, $\exists\, A_n \in \mathcal{M}$ such that $B \backslash E \subset A_n$, $\mu^*(A_n) \leq 1/n$. Then let $A = \cap_n A_n$, $\mu(A) = 0$, $B \backslash E \subset A$. Therefore $(B \backslash A) \subset E$ and $E \backslash (B \backslash A) \subset A$, $E \subset \overline{\mathcal{M}}$. Therefore $\mathcal{M}^* = \overline{\mathcal{M}}$. Obviously the measure on $\overline{\mathcal{M}}$ is the same as the completion of the measure.

(b) Denote the completion of $(\mu, \mathcal{M})$ with $(\hat{\mu}, \overline{\mathcal{M}})$, and the saturation of the completion $(\widetilde{\mu}, \widetilde{\mathcal{M}})$. First show that $\widetilde{\mathcal{M}} = \mathcal{M}^*$. Give any E that is locally $\hat{\mu}$-measurable, for any $F \subset X$ that $\mu^*(F) < \infty$, exists $A \in \mathcal{M}$ such that $F \subset A$ and $\mu^*(F) + \epsilon \geq \mu(A) = \hat{\mu}(A \cap (A \cap E)) + \hat{\mu}(A \cap (A \cap E)^c) \geq \mu^*(E \cap F) + \mu^*(E^c \cap F)$, therefore E is μ^*-measurable. Conversely, if E is μ^*-measurable, for any $A \in \hat{\mathcal{M}}$ such that $\hat{\mu}(A) < \infty$, obviously $A \in \mathcal{M}^*$, therefore $E \cap A \in \mathcal{M}^*$, $\mu^*(E \cap A) = \hat{\mu}(E \cap A) \leq \infty$. Then by (a), $E \cap A \in \overline{\mathcal{M}}$, therefore E is locally $\hat{\mu}$-measurable.

Now show that $\widetilde{\mu} = \overline{\mu}$. $\forall\, E \in \widetilde{\mathcal{M}}$, if E is in $\overline{\mathcal{M}}$, then $\widetilde{\mu}(E) = \overline{\mu}(E)$ since the extension is unique. If E is not in $\overline{\mathcal{M}}$, then $\widetilde{\mu}(E) = \infty$. If $\mu^*(E) < \infty$, then $E \in \overline{\mathcal{M}}$. Therefore $\widetilde{\mu} = \overline{\mu}$. ☐

Exercise 23 Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \leq a < b \leq \infty$.

(a) $\mathcal{A}$ is an algebra on $\mathbb{Q}$.

(b) The σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.

(c) Define μ_0 on $\mathcal{A}$ by $\mu_0(\varnothing) = 0$ and $\mu_0(A) = \infty$ for $A \neq \varnothing$. Then μ_0 is a premeasure on A, and there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is μ_0.

Proof. (a) Obviously $\mathbb{Q}$ and $\varnothing$ are in $\mathcal{A}$, and finite unions of elements in $\mathcal{A}$ are still in $\mathcal{A}$. Give $(a, b] \cap \mathbb{Q}$, its completion is $(-\infty, a] \cup (b, \infty] \cap \mathbb{Q}$ is still a finite union, therefore $\mathcal{A}$ is an algebra.

(b) Since for any $a \in \mathbb{Q}$, $\cap_{n=1}^{\infty}(a, a + 1/n] \cap \mathbb{Q} = \{a\}$ and $\mathbb{Q}$ is countable, any subset of $\mathbb{Q}$ may be generated by single point sets. Therefore $\mathcal{M}(\mathcal{A}) = \mathcal{P}(\mathbb{Q})$.

(c) It is easy to see that μ_0 is finitely additive. Two measures that agree with μ_0 when restricted to $\mathcal{A}$ may be given: (1) the counting measure; (2) the outer measure given by μ_0. They will produce different results on $\{0\}$. ☐

Exercise 24 Let μ be a finite measure on $(X, \mathcal{M})$, and let μ^* be the outer measure induced by μ. Suppose that $E \subset X$ satisfies $\mu^*(E) = \mu^*(X)$.

(a) If $A, B \in \mathcal{M}$ and $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.

(b) Let $\mathcal{M}_E = \{A \cap E : A \in \mathcal{M}\}$, and define the function ν on $\mathcal{M}_E$ defined by $\nu(A \cap E) = \mu(A)$. Then $\mathcal{M}_E$ is a σ-algebra on E and ν is a measure on $\mathcal{M}_E$.

Proof. (a) $\mu^*(X \backslash E) = 0$. Therefore $\mu(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(B \cap E) = \mu(B)$, and the reverse inequality is also true in the same sense. Therefore $\mu(A) = \mu(B)$.

(b) Obviously $\varnothing$ and E are in $\mathcal{M}_E$. For any $A \in \mathcal{M}$, the completion of $A \cap E$ in E is still in $\mathcal{M}_E$. $\mathcal{M}_E$ is also closed to countable unions since $\mathcal{M}$ is a σ-algebra. Give any countable collection of disjoint sets $\{A_j \cap E\}$, $\nu(\cup_j A_j \cap E) = \mu(\cup_j A_j)$. Let $B_n = A_n \backslash \cup_1^{n-1} A_n$, then $B_j \cap E = A_j \cap E$. Therefore $\mu(\cup_j A_j) = \sum_j \mu(B_j) = \sum_j \mu(A_j) = \sum_j \nu(A_j \cap E)$. ☐

Exercise 25 If $E \subset \mathbb{R}$, the following are equivalent:

(a) $E \in \mathcal{M}_m u$.

(b) $E = V \backslash N_1$ where V is a G_δ set and $\mu(N_1) = 0$.

(c) $E = H \cup N_2$ where H is an F_σ set an $\mu(N_2) = 0$.

Proof. Obviously (b) and (c) implies (a). Suppose $E \in \mathcal{M}_\mu$, if $\mu(E) < \infty$, give any positive integer n, according the previous proposition one may select an open set U_n and a compact set K_n such that the error of their measure is within $1/n$. Then by taking the countable union or intersetion one may find such H and V. If $\mu(E) = \infty$, let $E_j = E \cap (a_j, b_j]$. For any $\epsilon > 0$, for each j, one can find U_j such that $E_j \subset U_j$ and $\mu(U_j) \leq \mu(E_j) + 2^{-j}\epsilon$. Let $V = \cup_j U_j$, then $\mu(V \backslash E) = \sum_j \mu(U_j \backslash E_j) \leq \epsilon$. In the same sense one can find a countable union of compact sets, H, such that $\mu(E \backslash H) = 0$. ☐

Exercise 26 If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\epsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu(E \triangle A) < \epsilon$.

Proof. By theorem 1.18, give any $\epsilon > 0$ one can find a compact K and an open U such that $\mu(U) - \epsilon \leq \mu(E) \leq \mu(K) + \epsilon$. Therefore one can find finite union of open intervals $I = \cup_j I_j$ that $K \subset I \subset U$. Then $\mu(E \triangle I) = \mu(E \backslash I) + \mu(I \backslash E) \leq 2\mu(U \backslash K) = 2\epsilon$. □

Exercise 27 Denote the Cantor set C. Show that if $x, y \in C$ and $x < y$, there exists $z \notin C$ such that $x < z < y$.

Proof. If such z does not exist, then x, y must lie in the same interval, which implies $|x - y| < 3^{-n}$ for any n, thus $x = y$, contradiction. Therefore x and y must not lie in the same interval. Hence $\exists N$ such that x and y are seperated at the n-th iteration. Thus just pick any z in the middle third of the interval then $x < z < y$. □

Exercise 28 Let F be increasing and right continuous, and let μ_F be the assiciated measure. Then $\mu_F(\{a\}) = F(a) - F(a-)$, $\mu_F([a, b)) = F(b-) - F(a-)$, $\mu_F([a, b]) = F(b) - F(a-)$, and $\mu_F = F(b-) - F(a)$.

Proof. Since $\{a\} = \cap_n [a, a+1/n)$, $\mu_F(\{a\}) = \mu(\cap_n (a-1/n, a]) = \lim_{n\to\infty}(F(a) - F(a-1/n)) = F(a) - F(a-)$. Then $\mu_F([a, b)) = \mu_F((a, b]) + \mu(\{a\}) - \mu(\{b\}) = F(b-) - F(a-)$. The rest can be easily shown with the same argument. □

Exercise 29 Let E be a Lebesgue measurable set.

(a) If $E \subset N$ where N is the nonmeasurable set (taking one element of each equivalence class in $[0, 1)/\{x - y \in \mathbb{Q}\}$), then $m(E) = 0$.

(b) If $m(E) > 0$, then E contains a nonmeasurable set.

Proof. (a) Suppose $R = \mathbb{Q} \cap [0, 1)$. Take $E_r = \{x + r : x \in E \cap [0, 1 - r)\} \cup \{x + r - 1 : x \in E \cap [1 - r, 1)\}$. Then each E_r is measurable and a subset of $[0, 1)$. Therefore $1 = m([0, 1)) \geq m(\cup_r E_r) = \sum_r m(E_r) = \sum_r m(E)$, $m(E) = 0$. (b) Because of translation invariance it suffices to consider $E \subset [0, 1]$. Obviously $E = \cup_r E \cap N_r$. Then if each $E \cap N_r$ is measurable, $m(E) = \sum_r m(\cup_r(E \cap N_r)) = \sum_r m((E \cap N))$, therefore $m(E) = 0$, contradiction. □

Exercise 30 If $E \in \mathcal{L}$ and $m(E) > 0$, for any $\alpha < 1$ there is an interval I such that $m(E \cap I) > \alpha m(I)$.

Proof. Suppose that there exists an α such that for every open interval I, $m(E \cap I) \leq \alpha m(I)$. If E is bounded, then there exists a collection of disjoint open intervals such that $E \subset \cup_k I_k$ with $\sum_k m(I_k) \leq (1 + \epsilon)m(E)$ for any $\epsilon > 0$. Then $m(E) = m(\cup_k(E \cap I_k)) \leq \sum_k \alpha m(I_k) \leq \alpha(1 + \epsilon)(E)$, contradiction. If E is not bounded, by σ-finiteness, one may write $E = \cup_k E_k$ where $m(E_k) < \infty$ for each k. Take E_i such that $m(E_i) > 0$. Then for any $\alpha < 1$ there is an interval I such that $m(E \cap I) \geq m(E_i \cap I) > \alpha m(I)$. □

Exercise 31 If $E \in \mathcal{L}$ and $m(E) > 0$, the set $E - E = \{x - y : x, y \in E\}$ contains an interval centered at 0.

Proof. By exercise 30, there is an interval $I = (x_0 - \alpha, x_0 + \alpha)$ such that $m(E \cap I) > 3/4m(I)$. Suppose there is a δ such that $0 \leq \delta < a$ and $\delta \notin E - E$. Then for any pair $x, y \in E$, $x - y \neq \delta$. Let $E_1 = E \cap (x_0 - a, x_0]$, $E_2 = E \cap (x_0, x_0 + a)$. Then $\forall x \in E_1$, $x + \delta \in I$ but not in E. Therefore $E_1 + \delta \subset I \backslash E$. Similarly $E_2 - \delta \subset I \backslash E$. Then $m(E \cap I) \leq m(E_1) + m(E_2) \leq 2(m(I) - m(I \cap E)) < 2/3m(E \cap I)$, contradiction. Therefore $\delta \in E - E$ and $-\delta \in E - E$, $(-\alpha, \alpha) \subset E - E$. □

Exercise 33 There exists a Borel set $A \subset [0, 1]$ such that $0 < m(A \cap I) < m(I)$ for every subinterval I of $[0, 1]$.

Proof. Enumerate the subintervals of I with rational endpoints. Then construct a series of cantor sets. For I_1, split it into two disjoint intervals with finite measure. Then on each subinterval contruct a Cantor set K_1, K_1', both with finite measure. Next assume that $K_1, \cdots, K_n$ and $K_1', \cdots, K_n'$ are already given for $I_1, \cdots, I_n$. Let $L_n = (K_1 \cup \cdots \cup K_n) \cup (K_1' \cup \cdots \cup K_n')$, then L_n is compact and totally disconnected. Therefore $I_{n+1} \backslash L_n$ must contain some intervals, namely J_{n+1}. Then split J_{n+1} and construct K_{n+1} and K_{n+1}' on each subinterval. Let

$K = \cup_n K_n$ and then obviously K_n' is disjoint from K for any n. Since K is the union of some Cantor sets, it is a borel set.

Let I be some subinterval of $[0,1]$. Then there must be some I_n such that $I_n \subset I$. Therefore $K_n, K_n' \in I$. Then $0 < m(K_n \cap I_n) \leq m(K \cap I) < m(K \cap I) + m(K_n') \leq m(I)$. □

2 Chapter 2: Integration

Let the measurable space be $(X, \mathcal{M})$ for Exercise 1-7.

Exercise 1 Let $f : X \to \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Then f is measurable iff $f^{-1}(\{\pm\infty\}) \in \mathcal{M}$, and f is measurable on Y.

Proof. If f is measurable then $f^{-1}(\{\pm\infty\}) \in \mathcal{M}$. Give any borel set $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, $f^{-1}(B) \in \mathcal{M}$. Therefore $f^{-1}(B \cap \mathbb{R}) = f^{-1}(B) \cap Y \in \mathcal{M}$, f measurable on Y. Conversely, for any borel set $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, $f^{-1}(B) = f^{-1}((B \cap \mathbb{R}) \cup (B \cap \{\infty, -\infty\})) \in \mathcal{M}$, f measurable. □

Exercise 2 Suppose $f, g : X \to \overline{\mathbb{R}}$ are measurable.

(a) fg is measurable (where $0 \cdot (\pm\infty) = 0$).

(b) Fix $a \in \overline{\mathbb{R}}$ and define $h(x) = a$ if $f(x) = -g(x) = \pm\infty$ and $h(x) = f(x) + g(x)$ otherwise. Then h is measurable.

Proof. (a) It is easy to see that $(fg)^{-1}(\pm\infty) \in \mathcal{M}$. Consider fg on $Y = (fg)^{-1}(\mathbb{R})$. If both f and g are finite, then fg measurable on this domain Y_1. If one of the maps is infinite and the other map is zero, denote this domain with $Y_2 \in \mathcal{M}$. Y_2 is included in the inverse image of 0. Therefore fg is measurable on $Y_1 \cup Y_2 = Y$. Therefore fg is measurable on $\overline{\mathbb{R}}$ by exercise 1.

(b) Obviously $(f+g)^{-1}(\{\pm\infty\}) \in \mathcal{M}$. In the same sense consider $f+g$ on Y. If f and g are both finite, then $f+g$ is measurable on this domain Y_1. Otherwise these two maps produce infinity of different signs and included in the reverse image of a. Therefore $f+g$ is measurable on $\overline{\mathbb{R}}$. □

Exercise 3 If $\{f_n\}$ is a sequence of measurable functions on X, then $\{x : \lim f_n(x) \text{ exists}\}$ is a measurable set.

Proof. $\forall\, x \in X$, $\lim f_n(x)$ exists if and only if $g_3(x) = g_4(x)$, where $g_3(x) = \limsup f_n(x)$, $g_4(x) = \liminf f_n(x)$. Since f_n is measurable for each n, g_3 and g_4 are measurable, which implies $g_3 - g_4$ is also measurable on both $\mathbb{R}$ and $\overline{\mathbb{R}}$. Therefore $\{x : \lim f_n(x) \text{ exists}\} = (g_3 - g_4)^{-1}(\{0\}) \cup \{g_3^{-1}(\infty)\} \cap \{g_4^{-1}(\infty)\} \cup \{g_3^{-1}(-\infty)\} \cap \{g_4^{-1}(-\infty)\}$ is measurable. □

Exercise 4 If $f : X \to \overline{\mathbb{R}}$ and $f^{-1}((r, \infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

Proof. $\forall\, r \in \mathbb{R}$, by the definition of real numbers there is a cauthy sequence of increasing rational numbers q_n such that $\lim q_n = r$. Then $f^{-1}((r, \infty]) = f^{-1}(\cap_n (q_n, \infty]) = \cap_n f^{-1}((q_n, \infty]) \in \mathcal{M}$, f measurable. □

Exercise 5 If $X = A \cup B$ where $A, B \in \mathcal{M}$, a function f is measurable on X iff f measurable on both A and B.

Proof. Recall that f is measurable on $A \subset X$ if $f^{-1}(B) \cap A \in \mathcal{M}$ for any set B that is measurable. Therefore obviously f measurable on A and B. Conversely, give any measurable set M, then $f^{-1}(M) \cap A \in \mathcal{M}$, $f^{-1}(M) \cap B \in \mathcal{M}$. Then $f^{-1}(M) \in \mathcal{M}$. □

Exercise 6 The supremum of an uncountable family of measurable $\overline{\mathbb{R}}$-valued functions on X can fail to be measurable.

Solution. Consider any unmeasurable set Y (then it is uncountable), give $f_y = \chi_y$ for any $y \in Y$. Then $\sup_y f_y = \chi_Y$ is not measurable since Y is not measurable. □

Exercise 7 Suppose that for each $\alpha \in \mathbb{R}$ we are given a set $E_\alpha \in \mathcal{M}$ such that $E_\alpha \subset E_\beta$ whenever $\alpha < \beta$, $\cup_{\alpha\in\mathbb{R}} E_\alpha = X$, and $\cap_{\alpha\in\mathbb{R}} E_\alpha = \varnothing$. Then there is a measurable function $f : X \to \mathbb{R}$ such that $f(x) \le \alpha$ on E_α and $f(x) \ge \alpha$ on E_α^c for every α.

Solution. Take $f(x) = \inf\{q \in \mathbb{Q} : x \in E_q\}$. Then $\forall\, x \in E_\alpha$, for any rational q that $q > \alpha$, $x \in E_q$. Therefore $f(x) \le \alpha$. Similarly $\forall\, x \in E_\alpha^c$, $x \in E_q^c$ for any rational numbers $q \le a$, therefore $x \notin E_q$, x may only be in some E_q that $q > \alpha$, therefore $f(x) \ge \alpha$. Note that: (1) f is $\mathbb{R}$-valued since $\forall\, x \in X$, $x \in E_q$ for some rational q, therefore $f(x) \le q$; if $f(x) = -\infty$ then $x \in \cap_{\alpha\in\mathbb{R}} E_\alpha$ contradiction. (2) f is $\mathbb{R}$-measurable because $\forall\, \alpha \in \mathbb{R}$, $f^{-1}([\alpha, \infty)) = \cup_n f^{-1}([q_n, \infty)) = \cup_n \{x : f(x) \ge q_n\} = \cup_n E_q^c \in \mathcal{M}$ where q_n is some decreasing cauthy sequence of rationals that converges to α. □

Exercise 8 If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is borel measurable.

Proof. Without loss of generality, suppose f is increasing, then f^{-1} is also monotone increasing on $\mathrm{Im} f$. Thus $f^{-1}([a, \infty))$ must be some interval, therefore borel measurable. Hence f is borel measurable. □

Exercise 9 Let $f : [0, 1] \to [0, 1]$ be the cantor function, and let $g(x) = f(x) + x$.

(a) g is a bijection from $[0, 1]$ to $[0, 2]$, and $h = g^{-1}$ is continuous from $[0, 2]$ to $[0, 1]$.

(b) If C is the cantor set, $m(g(C)) = 1$.

(c) By Exercise 1.29, $g(C)$ contains a Lebesgue nonmeasurable set A. Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel measurable.

Proof. (a) Obviously g is monotone increasing and continuous, thus $g([0, 1]) = [0, 2]$, g is bijective. Therefore $\forall\, (a, b) \in [0, 1]$, $h^{-1}((a, b)) = g((a, b)) = (g(a), g(b))$, h is open.

(b) Recall $C = [0, 1]\backslash(\cup_k I_k)$. Since g is bijective and $\{I_k\}$ is pairwise disjoint, $g(C) = [0, 2]\backslash g(\cup_k I_k) = [0, 2]\backslash(\cup_k g(I_k))$. By the construction of f, f is constant on I_k. Thus $m(g(I_k)) = m(I_k)$. Therefore

$$m(g(C)) = m([0, 2]) - \sum_k m(I_k) = 1$$

(c) Since $B = g^{-1}(A) \subset g^{-1}(g(C)) = C$, B must be of zero measure because it is contained in some null sets. Since h is continuous hence borel measurable, if B is borel measurable then $A = h^{-1}(B)$ would be borel measurable, contradiction. □

Exercise 10 The following implications are valid iff the measure μ is complete.

(a) If f is measurable then $f = g$ μ-a.e., then g is measurable.

(b) If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e., then f is measurable.

Proof. (a) If μ is complete, then $g - f$ must be measurable since it is only non-zero on some null sets, therefore $g = g - f + f$ is Lebesgue measurable. Conversely, suppose any $N \subset E$ with E a null set. Then let $f = \chi_E$, $g = \chi_{E\backslash N}$. Then $f - g = \chi_N$ must be measurable. Therefore $N = (f - g)^{-1}(\{1\})$ is measurable.

(b) Since f_n is measurable for each n, $\lim f_n$ is measurable, and $\lim f_n = f$ μ-a.e.. If μ is complete, by (a) f is measurable. Conversely, suppose any subset N of a null set, take $f_n = 0$ for each n and $f = \chi_N$, then f is measurable, N must be measurable. □

Exercise 11 Suppose that f is a function on $\mathbb{R} \times \mathbb{R}^k$ such that $f(x, \cdot)$ is borel measurable for each $x \in \mathbb{R}$ and $f(\cdot, y)$ is continuous for each $y \in \mathbb{R}^k$. For $n \in \mathbb{N}$, define f_n as follows. For $i \in \mathbb{Z}$ let $a_i = i/n$, and for $a_i \le x \le a_{i+1}$ let

$$f_n(x, y) = \frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})}{a_{i+1} - a_i}$$

Then f_n is borel measurable on $\mathbb{R} \times \mathbb{R}^k$ and $f_n \to f$ pointwise; hence f is borel measurable on $\mathbb{R} \times \mathbb{R}^k$. Conclude by induction that every function on $\mathbb{R}^n$ that is continuous in each variable separately is Borel measurable.

Proof. Since $f(x,\cdot):\mathbb{R}^k\to\mathbb{R}$ and $x-a_i:\mathbb{R}\to\mathbb{R}$ is measurable, $f_n(x,y)$ is measurable. Now show that $f_n\to f$ pointwise. Since

$$\begin{aligned}|f-f_n| &= |f(x,y)-\frac{1}{a_{i+1}-a_i}f(a_{i+1},y)(x-a_i)-f(a_i,y)(x-a_{i+1})|\\ &= \frac{1}{a_{i+1}-a_i}|(f(x,y)-f(a_{i+1},y))(x-a_i)-(f(a_i,y)-f(x,y))(x-a_{i+1})|\end{aligned}$$

Suppose some $\epsilon>0$, then there is a open neighbourhood $B_\delta(x)$ such that $\forall\, x'\in B_\delta(x)$, $|f(x)-f(x_0)|<\epsilon$. Take n large enough such that $[a_i,a_{i+1}]$ is in that neighbourhood, then

$$|f-f_n|\le\frac{\epsilon}{a_{i+1}-a_i}|(a_{i+1}-a_i)|=\epsilon$$

Since $f_n\to f$, f is borel measurable on $\mathbb{R}\times\mathbb{R}^k$. If $f(x):\mathbb{R}\to\mathbb{R}$ is continuous, then it is measurable. Assume that if $f:\mathbb{R}^n\to\mathbb{R}$ is continuous with respect to each variable then it is measurable. Then suppose any function $g:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$. By previous exercise g is measurable. Therefore the proof is done by induction. □

Exercise 13 Suppose $\{f_n\}\subset L^+$, $f_n\to f$ pointwise, and $\int f=\lim\int f_n<\infty$. Then $\int_E f=\lim\int_E f_n$ forall $E\in\mathcal{M}$. However, this need not be true if $\int f=\lim\int f_n=\infty$.

Proof. By Fatou's lemma,

$$\int_E f=\int f\chi_E=\int\liminf f_n\chi_E\le\liminf\int f_n\chi_E=\liminf\int_E f_n$$

Conversely, write

$$\int f-\int_E f=\int_{E^c}f\le\liminf\int_{E^c}f_n=\liminf(\int f-\int_E f)=\int f-\limsup\int_E f$$

therefore $\limsup\int_E f_n\le\int_E f\le\liminf\int_E f_n$, $\lim\int_E f_n=\int_E f$, the proof is done. For counter-examples, just take $f_n=\chi_{[n,n+1]}+\chi_{(-\infty,0]}$ and $E=[0,\infty)$. □

Exercise 14 If $f\in L^+$, let $\lambda(E)=\int_E f\mathrm{d}\mu$ for $E\in\mathcal{M}$. Then λ is a measure on $\mathcal{M}$, and for any $g\in L^+$, $\int g\mathrm{d}\lambda=\int fg\mathrm{d}\mu$.

Proof. $\lambda(\varnothing)=0$. Suppose a collection of disjoint measurable sets $\{E_n\}$, then $\lambda(\cup_n E_n)=\int f\chi_{\cup_n E_n}\mathrm{d}\mu=\sum_n\int f\chi_{E_n}\mathrm{d}\mu=\sum_n\lambda(E_n)$, therefore λ is a measure.

Give $\phi=\sum_i a_i\chi_{E_i}$ a simple function. Then $\int\phi\mathrm{d}\lambda=\sum_i a_i\lambda(E_i)=\int f\sum_i a_i\chi_{E_i}\mathrm{d}\mu=\int f\phi\mathrm{d}\mu$. Now suppose $\{\phi_n\}$ an increasing collection of simple functions that $\phi_n\to g$. Then

$$\int g\mathrm{d}\lambda=\lim\int\phi_n\mathrm{d}\lambda=\lim\int f\phi_n\mathrm{d}\mu=\int fg\mathrm{d}\mu$$

□

Exercise 15 If $\{f_n\}\subset L^+$, f_n decreases pointwise to f, and $\int f_1<\infty$, then $\int f=\lim\int f_n$.

Proof. Obviously $\{f_1-f_n\}$ increases pointwise to $\{f_1-f\}$. Therefore by MCT,

$$\lim\int(f_1-f_n)=\int(f_1-f)$$

hence

$$\int f=\int f_1-\int(f_1-f)=\int f_1-\lim\int(f_1-f_n)=\lim f_n$$

where the last equality is because $\int(f_1-f_n)+\int f_n=\int f_1$. □

Exercise 16 If $f \in L^+$ and $\int f < \infty$, for every $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f > \int f - \epsilon$.

Proof. By the definition of integration, for every $\epsilon > 0$, there exists a simple function ϕ that $\int \phi > \int f - \epsilon$. Write $\phi = \sum_i a_i \chi_{E_i}$ with the standard representation (where $a_i \neq 0$ for each i). Let $E = \cup_i E_i$, then $\int_E f > \int \phi > \int f - \epsilon$. Now show that E is of finite measure. It is obvious that

$$\infty > \int \phi \geq \int \min\{a_i\}\chi_E = \min\{a_i\}\mu(E)$$

therefore $\mu(E) < \infty$. □

Exercise 17 Assume Fatou's Lemma and deduce the monotone convergence theorem.

Proof. Suppose $\{f_n\}$ is a sequence in L^+ such that $f_j \leq f_{j+1}$ for all j, and $f = \lim_{n\to\infty} f_n = \liminf f_n$, then by Fatou's lemma,

$$\int f = \int \liminf f_n \leq \liminf \int f_n$$

Conversely,

$$0 = \int \liminf (f - f_n) \leq \liminf \int (f - f_n) = \liminf (\int f - \int f_n) = \int f - \limsup \int f_n$$

where $\int (f - f_n) = \int f - \int f_n$ because of $\int (f - f_n + f_n) = \int f_n + \int (f - f_n) = \int f$. Thus $\int f = \lim \int f_n$. □

Exercise 18 Fatou's lemma remains valid if the hypothesis that $f_n \in L^+$ is replaced by the hypothesis that f_n is measurable and $f_n \geq -g$ where $g \in L^+ \cap L^1$.

Proof. Obviously $g_n = f_n + g \geq 0$. Then $\{g_n\}$ is a sequence in L^+. Therefore by Fatou's lemma,

$$\int \liminf g_n = \int \liminf f_n + \int g \leq \liminf \int f_n + \int g$$

therefore $\int \liminf f_n \leq \liminf \int f_n$. □

Exercise 19 Suppose $\{f_n\} \subset L^1(\mu)$ and $f_n \to f$ uniformly.

(a) If $\mu(X) < \infty$, then $f \in L^1(\mu)$ and $\int f_n \to \int f$.

(b) If $\mu(X) = \infty$, the conclusions of (a) can fail.

Proof. (a) Since $f_n \to f$ uniformly, $\exists N$ such that $\forall n \geq N$ and $\forall x \in X$, $|f(x) - f_n(x)| \leq 1$. Let $g(x) = |f_N(x)| + 1$, then $f_n \leq g$ for each n. Since

$$\int g = \int |f_N(x)| + 1 = \int f_N(x) + \mu(X) < \infty$$

by DCT $f \in L^1(\mu)$ and $\int f_n \to \int f$.

(b) Just take $f_n = (1/n)\chi_{[0,n)}$ □

Exercise 20 If $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e., $|f_n| \leq g_n$, and $\int g_n \to \int g$, then $\int f_n \to \int f$.

Proof. By taking real and imaginary parts, assume f_n and g_n are real. Then $f_n + g_n \geq 0$ and $g_n - f_n \geq 0$. By Fatou's Lemma,

$$\int (f + g) \leq \int \liminf (f_n + g_n) \leq \liminf \int (f_n + g_n) = \liminf \int f_n + \int g$$

$$\int (g - f) \leq \int \liminf (g_n - f_n) \leq \liminf \int (g_n - f_n) = \int g - \limsup \int f_n$$

thus $\int f_n \to \int f$. □

Exercise 21 Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f - f_n| \to 0$ iff $\int |f_n| \to \int |f|$.

Proof. Obviously

$$\left| \int |f| - \int |f_n| \right| = \left| \int |f| - |f_n| \right| \le \int |f - f_n| \to 0$$

Conversely, if $\int |f_n| \to \int |f|$, then by Exercise 20, $\int f_n \to \int f$. Thus $|\int f - \int f_n| = \int |f - f_n| \to 0$. ☐

Exercise 22 Let μ be a counting measure on $\mathbb{N}$. Interpret Fatou's lemma and the monotone and dominated convergence theorem as statements about infinite series.

Solution. Obviously the measure of a measurable function f on $(\mathbb{N}, \mu)$ is $\int f = \sum_n f(n) = \sum_n a_n$. Therefore by Fatou's lemma, suppose $\{a_{nk}\}$ a sequence of nonnegative numbers, then $\sum_k \liminf_n a_{nk} \le \liminf_n \sum_k a_{nk}$. By MCT, given a sequence of nonnegative numbers $\{a_{nk}\}$, if $a_{nk} \le a_{n+1,k}$ for every n and k, and $a_{nk} \to a_k$ for every k, then $\lim_n \sum_k a_{nk} = \sum_k a_k$. The DCT says that for any sequence of complex numbers $\{a_{nk}\}$ such that $|a_{nk}| \le |g_k|$ for each k, and $a_{nk} \to a_k$ for every k, then $\lim_n \sum_k a_{nk} = \sum_k a_k$. ☐

Exercise 25 Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $\{r_n\}_1^\infty$ be an enumeration of the rationals, and set $g(x) = \sum_1^\infty 2^{-n} f(x - r_n)$.

(a) $g \in L^1(m)$, and in particular $g < \infty$ a.e.

(b) g is discontinuous at every point and unbounded on every interval, and it remains so after any modification on a Lebesgue null set.

(c) $g^2 < \infty$ a.e., but g^2 not integrable on any interval.

Proof. (a) Observe

$$\int |g| = \int \sum_1^\infty \frac{f(x - r_n)}{2^n} = \sum_1^\infty \frac{1}{2^n} \int f(x - r_n) = \sum_1^\infty \frac{1}{2^{n-1}} < \infty$$

where by MCT,

$$\int f(x - r_n) = \lim_{t \to \infty} \int f(x - r_n) \chi_{(r_n + 1/t, r_n + 1)} = \lim_{t \to \infty} \int_{r_n}^{r_n + 1/t} (x - r_n)^{1/2} \mathrm{d}x = 2$$

therefore $g \in L^1(m)$, and obviously $g < \infty$ a.e.

(b) Suppose $x_0 \in \mathbb{R}$ with g continuous at x_0. Then obviously $g(x_0) < \infty$. For any $\epsilon > 0$ and $0 < \delta < 1$, there exists $r_n \in \mathbb{Q}$ such that $x_0 < r_n < x_0 + \delta$. Let $x' \in (r_n, x_0 + \delta)$ such that

$$g(x_0) + \epsilon < \frac{1}{2^n} f(x' - r_n)$$

then $g(x') \ge \frac{1}{2^n} f(x' - r_n) \ge g(x_0) + \epsilon$. Since δ is arbitrary, contradiction. For any interval $(a, b) \subset \mathbb{R}$, take $r_n \in (a, b)$. Then for any ϵ that is sufficiently large, $g(r_n + (\frac{1}{2^n}\epsilon)^2) \ge \epsilon$. Therefore $g(x)$ is unbounded on any interval. If after modification g is no longer unbounded on some interval, take this interval as the same interval (a, b). then $\exists \epsilon > 0$ such that $g(x - r_n) < \epsilon$ for all $x \in (a, b)$, then g is modified on at least $(r_n, r_n + (\frac{1}{2^n}\epsilon)^2)$ which has a non-zero measure, contradiction.

(c) By (a) it immediately follows that $g^2 < \infty$ a.e. For the second part, observe

$$\int g^2 \ge \int \sum_1^\infty \frac{f^2(x - r_n)}{4^n} = \sum_1^\infty \frac{1}{4^n} \int f^2(x - r_n) = \infty$$

where $\int f^2(x - r_n) = \infty$ follows the same argument as (a). ☐

Exercise 32 Suppose $\mu(X) < \infty$. If f and g are complex valued measurable functions on X, define

$$\rho(f,g) = \int \frac{|f-g|}{1+|f-g|}$$

Then ρ is a metric on the space of measurable functions if we identify functions that are equal a.e., and $f_n \to f$ w.r.t. this metric iff $f_n \to f$ in measure.

Proof. The triangle inequality is obvious since

$$\frac{|f-g|}{1+|f-g|} = 1 - \frac{1}{1+|f-g|}$$

is an increasing function of $|f-g|$. Suppose $\epsilon > 0$. If $f_n \to f$ in measure then for any $\eta > 0$, $\exists\, N$ such that $\forall\, n \geq N$,

$$\mu(E_n = \{x : |f_n(x) - f(x)| > \epsilon\}) < \eta$$

take $\eta = \epsilon$, then

$$\rho(f_n, f) = \int_{E_n} \frac{|f_n - f|}{1+|f_n-f|} + \int_{E_n^c} \frac{|f_n - f|}{1+|f_n-f|} \leq \mu(E_n) + \mu(X)\epsilon = \epsilon(1+\mu(X)) \to 0$$

Conversely suppose $\rho(f_n, f) \to 0$. Then $\forall\, \eta > 0$, $\exists\, N$ such that if $n \geq N$, $\rho(f_n, f) < \eta$. Consequently,

$$\frac{\epsilon}{1+\epsilon}\mu(E_n) \leq \int_{E_n} \frac{|f_n - f|}{1+|f_n-f|} \leq \eta$$

therefore $\forall\, t > 0$, take $\eta = \dfrac{\epsilon t}{1+\epsilon}$, then $\exists\, N$ such that $\mu(E_n) \leq \eta\dfrac{1+\epsilon}{\epsilon} = t$. □

Exercise 33 If $f_n \geq 0$ and $f_n \to f$ in measure, then $\int f \leq \liminf \int f_n$.

Proof. Recall that given a sequence of real numbers $\{a_n\}$, there exist a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to L$ for any $\liminf a_n \leq L \leq \limsup a_n$. Then there is a subsequence $\int f_{n_k}$ such that $\lim \int f_{n_k} = \liminf \int f_n$. Obviously $f_{n_k} \to f$ in measure, therefore there is a subsequence $f_{n_{k_i}}$ that converges to f a.e. Therefore by Fatou's Lemma,

$$\int f = \int \liminf_i f_{n_{k_i}} \leq \liminf_i \int f_{n_{k_i}} = \lim_k \int f_{n_k} = \liminf \int f_n$$

□

Exercise 34 Suppose $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure,

(1) $\int f = \lim \int f_n$,

(2) $f_n \to f$ in L^1.

Proof. (a) Since $f_n \to f$ in measure iff $\mathrm{Re}(f_n) \to f$ in measure and $\mathrm{Im}(f_n) \to f$ in measure, assume f_n and f are real-valued. Since $f_n \in L^1$ and there is a subsequence of f_n that converges to f a.e., $f \in L^1$. Since $g + f_n$ and $g - f_n$ are non-negative functions, the previous exercise implies that

$$\int g + \int f = \int \liminf(g+f_n) \leq \liminf \int (g+f_n) = \int g + \liminf \int f_n$$

$$\int g - \int f = \int \liminf(g-f_n) \leq \liminf \int (g-f_n) = \int g - \limsup \int f_n$$

therefore $\int f = \lim \int f_n$.

(b) Obviously $|f_n - f|$ converges to 0 in measure. Since $|f_n - f| \leq |f_n| + |f| \leq 2|g| \in L^1$, by (a), $\lim \int |f_n - f| = 0$, $f_n \to f$ in L^1. □

Exercise 35 $f_n \to f$ in measure iff for every $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \leq \epsilon$ for $n \geq N$.

Proof. For any $\epsilon, \eta > 0$, suppose $\eta < \epsilon$, then $\exists N$ such that $\forall n \geq N$, $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \leq \mu(\{x : |f_n(x) - f(x)| > \eta\}) < \eta$. The reverse direction is trivial. $\square$

Exercise 36 If $\mu(E_n) < \infty$ for $n \in \mathbb{N}$ and $\chi_{E_n} \to f \in L^1$, then f is a.e. equal to the characteristic function of a measurable set.

Proof. Since $\chi_{E_n} \to f$ in L^1, there exists a subsequence $\chi_{E_{n_k}} \to f$ a.e. Therefore there is a measurable function g such that $g = f$ a.e. Since f and g can only take values 0 or 1, $f = \chi_{g^{-1}\{1\}}$ a.e. $\square$

Exercise 37 Suppose that f_n and f are measurable compelx-valued functions and $\phi : \mathbb{C} \to \mathbb{C}$.

(a) If ϕ is continuous and $f_n \to f$ a.e., then $\phi \circ f_n \to \phi \circ f$ a.e.

(b) If ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly, or in measure, then $\phi \circ f_n \to \phi \circ f$, uniformly, almost uniformly, or in measure, respectively.

(c) There are counterexamples when the continuity assumptions on ϕ are not satisfied.

Proof. (a) Let $x \in X$ be a point where f_n converges to f. Then

$$\lim_{n\to\infty} \phi(f_n(x)) = \phi(\lim_{n\to\infty} f_n(x)) = \phi(f(x))$$

so $\phi \circ f_n \to \phi \circ f$ a.e.

(b) Suppose $f_n \to f$ uniformly, $\forall \epsilon > 0$, $\exists N$ such that $|f_n - f| < \epsilon$ for $n \geq N$. Since ϕ is also uniformly continuous, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|\phi(f_n) - \phi(f)| < \epsilon$ for any $|f_n - f| < \delta$. Therefore $\phi \circ f_n \to \phi \circ f$ uniformly. The same argument applys for the almost uniform case. If $f_n \to f$ in measure, since ϕ is uniformly continuous, $\exists \eta$,

$$\{x : |\phi(f_n(x)) - \phi(f(x))| < \epsilon\} \subset \{x : |f_n(x) - f(x)| < \eta\}$$

the proof is done since $\mu(\{x : |f_n(x) - f(x)| < \eta\}) \to 0$

(c) Give $f_n = e^{-n}$, $f_n \to f$ uniformly, suppose $\phi = \ln x$, then $\phi \circ f_n = -n$, which is anywhere divergent. $\square$

Exercise 38 Suppose $f_n \to f$ in measure and $g_n \to g$ in measure.

(a) $f_n + g_n \to f + g$ in measure.

(b) $f_n g_n \to fg$ in measure if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$.

Proof. (a) Let $\epsilon > 0$, then $\exists N_f, N_g$ such that $\mu(\{x : |f_n - f| \geq \epsilon/2\}) < \epsilon/2$ for $n > N_f$ and likewise for g. When n is large enough, since $|(f_n + g_n) - (f + g)| \leq |f_n - f| + |g_n - g|$,

$$\{x : |(f_n + g_n) - (f + g)| \geq \epsilon\} \subset \{x : |f_n - f| \geq \epsilon/2\} \cup \{x : |g_n - g| \geq \epsilon/2\}$$

therefore $\mu(\{x : |(f_n + g_n) - (f + g)| \geq \epsilon\}) \to 0$.

(b) Likewise define ϵ, N_f, N_g. Since $|f_n g_n - fg| \leq |f_n - f||g_n - g| + |f||g_n - g| + |g||f_n - f|$,

$$\{x : |fg - f_n g_n| > \epsilon\} \subset \{x : |f_n - f||g_n - g| > \epsilon/3\} \cup \{x : |f_n - f||g| > \epsilon/3\} \cup \{x : |f||g_n - g| > \epsilon/3\}$$

It is obvious that $\mu(\{x : |f_n - f||g_n - g| > \epsilon/3\}) \to 0$. To show $\mu(\{x : |f||g_n - g| > \epsilon/3\}) \to 0$, claim that for any $\eta > 0$, $\exists N \in \mathbb{N}$ such that $\mu(\{x : |f| > N\}) < \eta$. Let $E_n = \{x : |f| > n\}$, then E_n is a decreasing sequence of sets. Since $\mu(X) < \infty$, and $|f|$ can only take on finite values which implies $\cap_n E_n = \varnothing$, by convergence from below, $\mu(E_n) \to 0$, which verifies the claim. Since

$$\{x : |f||g_n - g| > \epsilon/3\} \subset \{x : |f| > N\} \cup \{x : |g_n - g| < \epsilon/3N\}$$

for each N, there is

$$\mu(\{x : |f_n - f||g_n - g| > \epsilon/3\}) \leq \mu(\{x : |f| > N\}) + \mu(\{x : |g_n - g| > \epsilon/3N\})$$

therefore $\forall \nu > 0$, take N and n such that $\mu(\{x : |f| > N\}) < \nu/2$ and $\mu(\{x : |g_n - g| > \epsilon/3N\} < \nu/2$, it can be seen that $\mu(\{x : |f||g_n - g| > \epsilon/3\}) \to 0$, similarly $\mu(\{x : |g||f_n - f| > \epsilon/3\}) \to 0$, the proof is done. $\square$

Exercise 39 If $f_n \to f$ almost uniformly, then $f_n \to f$ a.e. and in measure.

Proof. Since $f_n \to f$ almost uniformly, $\forall n \in \mathbb{N}$, $\exists E_n \subset X$ such that $\mu(E_n) < 1/n$ and $f_n \to f$ uniformly on E_n^c. Then obviously $E = \cap_n E_n$ has zero measure by continuity from below, and $f_n \to f$ on E^c. Therefore $f_n \to f$ a.e.

$\forall \epsilon > 0$, take $E \subset X$ such that $f_n \to f$ uniformly on E^c and $\mu(E) < \epsilon$. Then $\forall \eta > 0$, $\exists N$ such that if $n > N$

$$\{x : |f_n - f| > \eta\} \subset E$$

therefore $f_n \to f$ in measure. □

Exercise 40 In Egoroff's theorem, the hypothesis "$\mu(X) < \infty$" can be replaced by "$|f_n| \leq g$ for all n, where $g \in L^1(\mu)$".

Proof. Without loss of generality, assume $f_n \to f$ for all $x \in X$. For $k, n \in \mathbb{N}$, let

$$E_n(k) = \bigcup_{m=n}^{\infty} \{x : |f_m - f| \geq k^{-1}\}$$

then for fixed k, E_n is a decreasing sequence. For $x \in X$, if $x \in E_1(k)$, then $\exists m$ such that $|f_m - f| \geq 1/k$. Therefore $1/k \leq |f_m + f| \leq 2g$, $\int 1/2k\chi_{E_1(k)} = 1/2k\mu(E_1(k)) \leq \int g$. Since $g \in L^1$, $\mu(E_1(k)) < \infty$. Therefore by continuity from below, $\mu(E_n(k)) \to 0$. Given $\epsilon > 0$ and $k \in \mathbb{N}$, choose n_k so large that $\mu(E_{n_k}(k)) \leq \epsilon 2^{-k}$, and let $E = \cup_k E_{n_k}(k)$. Then $\mu(E) \leq \epsilon$, and $|f_n - f| \leq 1/k$ for $n > n_k$ and $x \in E^c$. □

Exercise 41 If μ is σ-finite and $f_n \to f$ a.e., there exist measurable $E_1, E_2, \cdots \subset X$ such that $\mu((\cup_1^\infty E_j)^c) = 0$ and $f_n \to f$ uniformly on each E_j.

Proof. Suppose $\mu(X) < \infty$, then by Egoroff's theorem, for each $k \in \mathbb{N}$, $\exists E_k$ such that $\mu(E_k^c) < 1/k$ and $f_n \to f$ uniformly on E_k. Let $F_n = \cup_1^n E_k$, then F_n^c is a decreasing sequence, therefore

$$\mu\left(\left(\bigcup_1^\infty E_j\right)^c\right) = \mu\left(\left(\bigcup_1^\infty F_j\right)^c\right) = \mu\left(\bigcap_1^\infty F_j\right) = 0$$

and $f_n \to f$ uniformly on each E_j.

Since μ is σ-finite, $X = X_1 \cup X_2 \cdots$ each with finite measure. Therefore for each i, there exists $\{E_k^i\}$ such that $\mu(X_i \backslash (\cup_k E_k^i)) = 0$ and $f_n \to f$ uniformly on each E_k^i. Since

$$\mu\left(\left(\bigcup_{i,k} E_k^i\right)^c\right) \leq \mu\left(\bigcup_i \left(X_i \backslash \bigcup_k E_k^i\right)\right) = 0$$

$\{E_k^i\}$ gives the desired sequence. □

Exercise 42 Let μ be the counting measure on $\mathbb{N}$. Then $f_n \to f$ in measure iff $f_n \to f$ uniformly.

Proof. Suppose $f_n \to f$ in measure. Then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n > N$,

$$\mu(\{x : |f_n - f| > \epsilon\}) < 1/2$$

therefore $|f_n - f| < \epsilon$ for each $x \in \mathbb{N}$, hence $f_n \to f$ uniformly. Conversely, if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n > N$, $|f_n - f| < \epsilon$ for each $x \in \mathbb{N}$, then $\mu(\{x : |f_n - f| > \epsilon\}) = 0$. □

Exercise 44 If $f : [a,b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subset [a,b]$ such that $\mu(E^c) < \epsilon$ and $f|_E$ is continuous.

Proof. For each $n \in \mathbb{N}$, let $E_n = f^{-1}(B_n(0))$. Then

$$\lim \mu(E_n) = \mu(\cup_n E_n) = \mu([a,b])$$

therefore $\exists\, m \in \mathbb{N}$ such that $\mu([a,b]) - \mu(E_m) \leq \epsilon/3$. Then $|f\chi_{E_m}| \leq m\chi_{[a,b]}$, thus $g \in L^1$. Hence by theorem 2.26 there is a sequence of continuous functions $g_j \to f\chi_{E_m}$. By corollary 2.32, there is a subsequence $g_{j_i} \to f\chi_{E_m}$ a.e. By Egoroff's theorem, there exists $F \subset E_m$ such that $g_{j_i} \to f\chi_{E_m}$ uniformly on $E_m \backslash F$ and $\mu(F) < \epsilon/3$. By theorem 1.18, there exists a compact set E such that $E \subset E_m \backslash F$ and $\mu(E) > \mu(E_m \backslash F) + \epsilon/3$. Therefore $f\chi_E$ is continuous, and

$$\mu(E^c) = \mu(E_m^c) + \mu(E_m \backslash E) \leq \epsilon/3 + \mu(E_m \backslash F) + \mu(E_m \backslash F \backslash E) \leq \epsilon$$

□

Exercise 45 If $(X_j, \mathcal{M}_j)$ is a measurable space for $j = 1,2,3$, then $\bigotimes_1^3 \mathcal{M}_j = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$. Moreover, if μ_j is a σ-finite measure on $(X_j, \mathcal{M}_j)$, then $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$

Proof. $(\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$ is generated by $\mathcal{E} = \{(E_1 \times E_2) \times E_3 : E_j \in \mathcal{M}_j\}$. By the natural identification, one takes $(X_1 \times X_2) \times X_3 = X_1 \times X_2 \times X_3$. Thus $\mathcal{E} = \{E_1 \times E_2 \times E_3 : E_j \in \mathcal{M}_j\}$, which generates $\bigotimes_1^3 \mathcal{M}_j$.

Suppose μ_1, μ_2, μ_3 are σ-finite. Then on the algebra $\mathcal{A}$ of rectangles,

$$(\mu_1 \times \mu_2) \times \mu_3((E_1 \times E_2) \times E_3) = \mu_1(E_1)\mu_2(E_2)\mu_3(E_3) = \mu_1 \times \mu_2 \times \mu_3(E_1 \times E_2 \times E_3)$$

since $(\mu_1 \times \mu_2) \times \mu_3$ and $\mu_1 \times \mu_2 \times \mu_3$ are both σ-finite measures and they agree on $\mathcal{A}$, they are equal by the uniqueness assertion in theorem 1.14. □

Exercise 46 Let $X = Y = [0,1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, μ is the Lebesgue measure, and ν is the counting measure. If $D = \{(x,x) : x \in [0,1]\}$ is the diagonal in $X \times Y$, then $\int\int \chi_D \mathrm{d}\mu\mathrm{d}\nu$, $\int\int \chi_D \mathrm{d}\nu\mathrm{d}\mu$, and $\int \chi_D \mathrm{d}(\mu \times \nu)$ are all unequal.

Proof. Obviously,

$$\int\int \chi_D \mathrm{d}\mu\mathrm{d}\nu = \int \left[\int \chi_D^y \mathrm{d}\mu\right] \mathrm{d}\nu = 0$$

$$\int\int \chi_D \mathrm{d}\nu\mathrm{d}\mu = \int \left[\int \chi_D^x \mathrm{d}\nu\right] \mathrm{d}\mu = \int \mathrm{d}\mu = 1$$

By definition,

$$\int \chi_D \mathrm{d}(\mu \times \nu) = \inf\{\sum_{n=1}^{\infty} \mu(A_j)\nu(B_j) : D \subset \cup_j (A_j \times B_j) \text{ where } A_j \times B_j \text{ are disjoint rectangles}\}$$

Suppose such sequence $A_j \times B_j$ that covers D. Then $[0,1] \subset \cup_j (A_j \cap B_j)$. Therefore $\mu(A_n \cap B_n) > 0$ for some n. Then $\mu(A_n) > 0$, and $\nu(B_n) = \infty$. Therefore the integral is ∞. □

Exercise 48 Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, μ and ν are the counting measure. Define $f(m,n) = 1$ if $m = n$ and $f(m,n) = -1$ if $m = n+1$, and $f(m,n) = 0$ otherwise. Then $\int |f| \mathrm{d}(\mu \times \nu) = \infty$, and $\int\int f \mathrm{d}\mu\mathrm{d}\nu$ and $\int\int f \mathrm{d}\nu\mathrm{d}\mu$ exist and are unequal.

Proof.

$$\int \left[\int f^y \mathrm{d}\mu\right] \mathrm{d}\nu = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} f(n,j) = 0$$

$$\int \left[\int f_x \mathrm{d}\nu\right] \mathrm{d}\mu = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} f(n,j) = 1$$

Let $E_1 = \{(n,n) : n \in \mathbb{N}\}$ and $E_2 = \{(n,n+1) : n \in \mathbb{N}\}$, then $|f(x)| = 1$ and non-zero iff $x \in E_1 \cup E_2$. Thus

$$\int |f| \mathrm{d}(\mu \times \nu) = (\mu \times \nu)(E_1) + (\mu \times \nu)(E_2) = \infty$$

since E_1 and E_2 are not finite. □

Exercise 49 Prove Theorem 2.39 by using Theorem 2.37 and Proposition 2.12 together with the following lemmas.

(a) If $E \in \mathcal{M} \otimes \mathcal{N}$ and $\mu \times \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y.

(b) If f is $\mathcal{L}$-measurable and $f = 0$ λ-a.e., then f_x and f^y are integrable for a.e. x and y, and $\int f_x \mathrm{d}\nu = \int f^y \mathrm{d}\mu = 0$ for a.e. x and y.

Proof. (a) Since μ and ν are σ-finite,

$$0 = (\mu \times \nu)(E) = \int \mu(E^y)\mathrm{d}\nu(y) = \int \nu(E_x)\mathrm{d}\mu(x)$$

therefore $\nu(E_x) = \mu(E^y) = 0$ a.e. x and y.

(b) Let $E \subset X \times Y$ be the null set such that $f(x,y) = 0$ for all $(x,y) \notin E$. Since λ is the completion of $\mu \times \nu$, there is a set $E' \in \mathcal{M} \otimes \mathcal{N}$ such that $E \subset E'$ and $(\mu \times \nu)(E') = 0$. Therefore

$$0 = (\mu \times \nu)(E') = \int \mu(E'^y)\mathrm{d}\nu(y) = \int \nu(E'_x)\mathrm{d}\mu(x)$$

thus $\nu(E'_x) = 0$ and $\mu(E'^y) = 0$ a.e. Since μ and ν are complete, $\mu(E_x) = 0$ and $\nu(E^y) = 0$ a.e. Therefore $f_x = 0$ and $f^y = 0$ a.e. Hence f_x and f^y are measurable and integrable a.e. with $\int f_x \mathrm{d}\nu = \int f^y \mathrm{d}\mu = 0$.

Now assume f is $\mathcal{L}$-measurable. There exists an $(\mathcal{M} \otimes N)$-measurable function g such that $f = g$ λ-a.e. If $f \geq 0$, then $g \geq 0$ a.e. Without the loss of generality assume $g \geq 0$, by Tonelli's theorem, $x \mapsto \int g_x \mathrm{d}\nu$ and $y \mapsto \int g^y \mathrm{d}\mu$ are non-negative and $(M \otimes N)$-measurable with

$$\int g \mathrm{d}\lambda = \int\int g(x,y)\mathrm{d}\mu(x)\mathrm{d}\nu(y) = \int\int g(x,y)\mathrm{d}\nu(y)\mathrm{d}\mu(x) \quad (*)$$

Since $g = f$ λ-a.e., if $f \in L^1(\lambda)$ then $g \in L^1(\mu \times \nu)$. By Fubini's theorem, this implies that $g_x \in L^1(\nu)$, $g_y \in L^1(\mu)$, $x \mapsto \int g_x \mathrm{d}\nu \in L^1(\mu)$ and $y \mapsto g_y \mathrm{d}(\mu) \in L^1(\nu)$ a.e. x and y, and $(*)$ holds.

Apply (b) to $f - g$, therefore $f_x \in L^1(\nu)$ and $f^y \in L^1(\mu)$ a.e. x and y provided that $f \in L^1(\lambda)$. In either cases, $\int g_x \mathrm{d}\nu = \int f_x \mathrm{d}\nu$ a.e. x, therefore $\int f_x \mathrm{d}\nu$ is measurable and the same holds for y. Because $f = g$ a.e.,

$$\begin{aligned}\int f \mathrm{d}\lambda &= \int g \mathrm{d}\lambda \\ &= \int\int g(x,y)\mathrm{d}\mu(x)\mathrm{d}\nu(y) = \int\int g(x,y)\mathrm{d}\nu(y)\mathrm{d}\mu(x) \\ &= \int\int f(x,y)\mathrm{d}\mu(x)\mathrm{d}\nu(y) = \int\int f(x,y)\mathrm{d}\nu(y)\mathrm{d}\mu(x)\end{aligned}$$

□

Exercise 50 Suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and $f \in L^+(X)$. Let

$$G_f = \{(x,y) \in X \times [0,\infty] : y \leq f(x)\}$$

then G_f is $\mathcal{M} \times \mathcal{B}_{\mathbb{R}}$-measurable and $\mu \times m(G_f) = \int f \mathrm{d}\mu$; the same is also true if the inequality $y \leq f(x)$ in the definition of G_f is replaced by $y < f(x)$.

Proof. Since $g = (x,y) \mapsto (f(x) - y) = ((s,t) \mapsto (s-t)) \circ ((x,y) \mapsto (f(x),y))$, $G_f = g^{-1}([0,\infty))$ is measurable. Then

$$(\mu \times m)(G_f) = \int m((G_f)_x)\mathrm{d}\mu(x) = \int f \mathrm{d}\mu$$

□

Exercise 51 Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be arbitrary measure spaces.

(a) If $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable, $g : Y \to \mathbb{C}$ is $\mathcal{N}$-measurable, and $h(x, y) = f(x)g(y)$, then h is $\mathcal{M} \otimes \mathcal{N}$-measurable.

(b) If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and $\int h\mathrm{d}(\mu \times \nu) = (\int f\mathrm{d}\mu)(\int g\mathrm{d}\nu)$

Proof. (a) Since $f(x)$ and $g(y)$ are $\mathcal{M} \otimes \mathcal{N}$-measurable, $h = fg$ is also measurable.

(b) Suppose $f \geq 0$ and $g \geq 0$. Then there exist increasing sequences ϕ_n and ψ_n of non-negative simple functions that converges to f and g respectively. Then $\phi_n\psi_n \to h$ pointwise. Suppose $\phi_n = \sum_i^k a_i\chi_{A_i}$, $\psi_n = \sum_j^l b_j\chi_{B_j}$. Then

$$\int \phi_n\psi_n = \sum_i^k \sum_j^l a_i b_j (\mu \times \nu)(A_i \times B_j) = \left(\sum_i^k a_i\mu(A_i)\right)\left(\sum_j^l b_j\nu(B_j)\right) = \int \phi_n \cdot \int \psi_n$$

therefore it is true for positive functions. For any complex function g, just decompose it into $u = \mathrm{Re}g$, $v = \mathrm{Im}g$ then u^+, u^-, v^+, v^-. Apply the above formula repeatedly, the proof is complete. ☐

Exercise 52 The Fubini-Tonelli theorem is valid when $(X, \mathcal{M}, \mu)$ is an arbitrary measure space and Y is a countable sets, $\mathcal{N} = \mathcal{P}(Y)$, and ν is counting measure on Y.

Proof. If $f \in L^+(X \times Y)$, since ν is the counting measure, identify it with $\mathbb{N}$. Then

$$\int_X \int_{\mathbb{N}} f_x(n)\mathrm{d}\nu\mathrm{d}\mu = \int_X \left(\sum_1^\infty f_x(n)\right)\mathrm{d}\mu = \int_{\mathbb{N}}\int_X f^n(x)\mathrm{d}\mu\mathrm{d}\nu = \sum_1^\infty \left(\int_X f^n(x)\mathrm{d}\mu\right) = \int f\mathrm{d}(\mu \times \nu)$$

therefore Fubini-Tonelli theorem is true. ☐

3 Chapter 3: Signed Measures and Differentiation

Exercise 1 Prove Proposition 3.1.

Proof. Suppose $\{E_j\}$ an increasing sequence, $F_j = E_j \setminus \cup_1^{j-1} E_i$, since $\mu(E_j) = \sum_{k=1}^n \mu(F_k)$,

$$\mu(\cup_j E_j) = \mu(\cup_j F_j) = \sum_j \mu(F_j) = \lim \mu(E_j)$$

Suppose $\{E_j\}$ an decreasing sequence, since $\mu(E_1) < \infty$,

$$\mu(\cap_j E_j) = \mu(E_1 \setminus (E_1 \setminus \cap_j E_j)) = \mu(E_1) - \mu(\cup_j(E_1 \setminus E_j)) = \mu(E_1) - \lim(\mu(E_1) - \mu(E_j)) = \lim \mu(E_j)$$

☐

Exercise 3 Let ν be a signed measure on $(X, \mathcal{M})$.

(a) $L^1(\nu) = L^1(|\nu|)$

(b) If $f \in L^1(\nu)$, $|\int f d\nu| \leq \int |f| d|\nu|$

(c) If $E \in \mathcal{M}$, $|\nu|(E) = \sup\{|\int_E f d\nu| : |f| \leq 1\}$.

Proof. (a) Let $\phi \in L^1$ be a simple function, and write $\phi = \sum_{i=1}^n a_i\chi_{E_i}$, then

$$\int \phi d|\nu| = \sum_{i=1}^n a_i|\nu|(E_i) = \sum_{i=1}^n a_i(\nu^+(E_i) + \nu^-(E_i)) = \int \phi d\nu^+ + \int \phi d\nu^-$$

since for any $f \in L^1(\nu)$, $f \in L^1(\nu^+) \cap L^1(\nu^-)$, thus

$$\int |f| d|v| = \left\{\int \phi d|\nu| : \phi \in L^+ \text{ simple, } \phi \leq |f|\right\} = \int |f| dv^+ + \int |f| dv^- \leq \infty$$

hence $L^1(\nu) \subset L^1(|\nu|)$. The converse is obviously true.

(b)

$$\left|\int f d\nu\right| = \left|\int f d\nu^+ - \int f d\nu^-\right| = \left|\int f d\nu^+ - \int f d\nu^-\right| \le \int |f| dv^+ + \int |f| dv^- = \int |f| d|\nu|$$

(c) Suppose $g = \chi_B - \chi_A$, where A and B are the hahn decomposition of ν. Then

$$\int_E g d\nu = \int (\chi_B - \chi_A)\chi_E d\nu = \int \chi_{B\cap E} d\nu^+ + \int \chi_{A\cap E}\nu^- = \nu^+(E) + \nu^-(E) = |\nu|(E)$$

If $|\nu|(E) = \infty$, the proof is done. Otherwise assume that $|\nu|(E) < \infty$, and let f be a measurable function with $|f| \le 1$. Then $|\int_E f d\nu| \le \int_E |f| d|\nu| \le |\nu|(E)$. Therefore

$$|\nu|(E) \le \left\{ |\int_E f d\nu| : |f| \le 1 \right\} \le |\nu|(E)$$

the proof is complete. □

Exercise 4 If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \ge \nu^+$ and $\mu \ge \nu^-$.

Proof. Suppose hahn decomposition A, B for ν, then $\forall\, E \in \mathcal{M}$,

$$\lambda(E) \ge \lambda(E \cap B) \ge \nu(E \cap B) = \nu^+(E \cap B) \ge \nu(E)$$

the same argument goes for $\mu \ge \nu^-$. □

Exercise 5 If ν_1, ν_2 are signed measures that both omit the value $+\infty$ or $-\infty$, then $|\nu_1 + \nu_2| \le |\nu_1| + |\nu_2|$.

Proof. Obviously $\nu_1 + \nu_2$ is still a signed measure, and $\nu_1 + \nu_2 = (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)$. By exercise 4, $(\nu_1^+ + \nu_2^+) \ge (\nu_1 + \nu_2)^+$ and $(\nu_1^- + \nu_2^-) \ge (\nu_1 + \nu_2)^-$. Therefore

$$|\nu_1 + \nu_2| = (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \le (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) = |\nu_1| + |\nu_2|$$

□

Exercise 6 Suppose $\nu(E) = \int_E f d\mu$ where μ is a positive measure and f is an extended μ-integrable function. Describe the Hahn decompositions of ν and the positive, negative, and total variations of ν in terms of f and μ.

Solution. $P = \{x : f(x) \ge 0\}$, $N = \{x : f(x) < 0\}$. $\nu^+ = \int_{E\cap P} f d\nu$, $\nu^- = -\int_{E\cap N} f d\nu$, $|\nu| = \nu^+ + \nu^-$. □

Exercise 7 Suppose that ν is a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$.

(a) $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$ and $\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\}$.

(b) $|\nu|(E) = \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \cdots, E_n$ are disjoint, and $\cup_1^n E_j = E\}$.

Proof. (a) Let A and B be the hahn decomposition. Then

$$\nu^+(E) = \nu^+(E \cap P) \le \sup\{\nu(F) : F \subset E\}$$

moreover, if $F \subset E$, then

$$\nu(F) = \nu^+(F) \le \nu^+(E)$$

therefore

$$\nu^+(E) = \sup\{\nu(F) : F \subset E\}$$

the similar argument works for $v^-(E)$.

(b) Denote RHS with t.

$$|\nu|(E) = |\nu(E \cap A)| + |\nu(E \cap B)| \le t$$

moreover,

$$\sum_1^n |\nu(E_j)| \le \sum_1^n (\nu^+(E_j) + \nu^-(E_j)) = \nu^+(E) + \nu^-(E) = |\nu|(E)$$

the proof is complete. □

Exercise 8 $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Proof. Suppose $\mu(E) = 0$, then $|\nu|(E) = 0$ iff $\nu^+(E) = \nu^-(E) = 0$. If $\nu \ll \mu$, since $\forall F \in \mathcal{M}$ that is contained in E, $\nu(F) = \mu(F) = 0$, by exercise 2, $|\nu|(E) = 0$. The converse is trivial. $\square$

Exercise 9 Suppose $\{v_j\}$ is a sequence of positive measures. If $\nu_j \perp \mu$ for all j, then $\sum_1^\infty \nu_j \perp \mu$; and if $\nu_j \ll \mu$ for all j, then $\sum_1^\infty \nu_j \ll \mu$.

Proof. The second part is trivial by countable additivity. For the first part, denote E_j the ν_j-null set and E_j^c the μ-null set, then $\cap_j E_j$ is $\sum_1^\infty \nu_j$-null and $(\cap_j E_j)^c$ is μ-null. $\square$

Exercise 10 Theorem 3.5 may fail when ν is not finite.

Solution. Take $d\nu(x) = dx/x$ and $d\mu(x) = dx$ on $(0, 1)$. Then obviously $\nu \ll \mu$, but consider $E_n = (0, 1/n)$, obviously $\nu(E_n) > 1$. $\square$

Exercise 11 Let μ be a positive measure. A collection of functions $\{f_\alpha\}_{\alpha\in A} \subset L^1(\mu)$ is called uniformly integrable if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\int_E f_\alpha d\mu| < \epsilon$ for all $\alpha \in A$ whenever $\mu(E) < \delta$.

(a) Any finite subset of $L^1(\mu)$ is uniformly integrable.

(b) If $\{f_n\}$ is a sequence in $L^1(\mu)$ that converges in the L^1 metric to $f \in L^1(\mu)$, then $\{f_n\}$ is uniformly integrable.

Proof. (a) Since $f \in L^1(\mu)$, the finite signed measure $E \mapsto \int_E f d\mu$ is absolutely continuous with respect to μ. Therefore for any $\epsilon > 0$, $\exists\, \delta_\alpha$ such that $|\int_E f_\alpha d\mu| < \epsilon$ when $\mu(E) < \delta_\alpha$. Just take $\delta = \min\{\delta_\alpha\} > 0$.

(b) For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\int |f_n - f| d\mu < \epsilon/2$ for any $n \geq N$. Let $I = \{0, 1, 2, \cdots, N\}$ (with $f_0 = f$), then $\{f_i\}_{i\in I}$ is uniformly integrable. Therefore $\exists\, \delta > 0$ such that $|\int_E f_i d\mu| < \epsilon/2$ for any $i \in I$ with $\mu(E) < \delta$. Then for $i \in \mathbb{N}\backslash I$,

$$\left|\int_E f_n d\mu\right| = \left|\int_E (f_n - f) d\mu + \int_E f d\mu\right| \leq \left|\int_E (f_n - f) d\mu\right| + \left|\int_E f d\mu\right| \leq \epsilon$$

$\square$

Exercise 12 For $j = 1, 2$, let μ_j, ν_j be σ-finite measures on $(X_j, \mathcal{M}_j)$ such that $\nu_j \ll \mu_j$. Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$, and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2)$$

Proof. If $(\mu_1 \times \mu_2)(E) = 0$, then

$$0 = \int \mu_2(E^{x_1}) d\mu_1(x_1)$$

therefore $\mu_2(E^{x_1})$ is μ_1 a.e. and hence $\nu_2(E^{x_1})$ is ν_1 a.e., then

$$(\nu_1 \times \nu_2)(E) = \int \nu_2(E^{x_1}) d\nu_1(x_1) = 0$$

The second part is verified by

$$\begin{aligned}(\nu_1 \times \nu_2)(E) &= \int f\chi_E d(\mu_1 \times \mu_2) = \int \left[\int f\chi_E d\mu_2(x_2)\right] d\mu_1(x_1) \\ &= \int \nu_2(E^{x_1}) d\nu_1(x_1) = \int \left[\int_E \frac{d\nu_2}{d\mu_2}(x_2) d\mu_2(x_2)\right] d\nu_1(x_1) \\ &= \int \left[\int \chi_E \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2) d\mu_2(x_2)\right] d\mu_1(x_1)\end{aligned}$$

$\square$

Exercise 13 Let $X = [0,1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, m is the Lebesgue measure, and μ is the counting measure on $\mathcal{M}$.
(a) $m \ll \mu$ but $dm \neq f d\mu$ for any f.
(b) μ has no Lebesgue decomposition with respect to m.

Proof. (a) The first part is trivial. Suppose there exists such f, then $m(\{x\}) = f(x) = 0$, therefore $f = 0$ and

$$1 = m(X) = \int_X f d\mu = 0$$

contradiction.

(b) Suppose that μ has a Lebesgue decomposition $\lambda + \rho$ with respect to m, with $\lambda \perp m$ and $\rho \ll m$. Then $\rho(\{x\}) = 0$, and $\lambda(\{x\}) = 1$. Suppose $X = A \sqcup B$ the Lebesgue decomposition with $\lambda(A) = m(B) = 0$. Then $A = \varnothing$, then $m(B) = m(X) = 1$, contradiction. □

Exercise 16 Suppose that μ, ν are σ-finite measures on $(X, \mathcal{M})$ with $\nu \ll \mu$, and let $\lambda = \mu + \nu$. If $f = d\nu/d\lambda$, then $0 \leq f < 1$ ν-a.e. and $d\nu/d\mu = f/(1-f)$.

Proof. Let $E_n = \{x : f(x) < -1/n\}$. Therefore

$$-n^{-1}\lambda(E_n) \geq \int_{E_n} f d\lambda = \nu(E_n) \geq 0$$

and hence $\mu(E_n) \leq \lambda(E_n) = 0$. It follows that $\mu(\cup_1^\infty E_n) = 0$, so $f \geq 0$ μ-a.e. Set $F = \{x : f(x) \geq 1\}$. Since ν is σ-finite, there is a sequence F_n of subsets of F which cover F such that $\nu(F_n) < \infty$ for each n. Because

$$\nu(F_n) = \int_{F_n} f d\lambda \geq \int_{F_n} 1 d\lambda = \lambda(F_n) = \mu(F_n) + \nu(F_n)$$

$\mu(F_n) = 0$. Thus $\mu(F) = 0$ and $f < 1$ μ-a.e. Therefore $f, 1-f \in L^+$, so for each $E \in \mathcal{M}$,

$$\int_E (1-f) d\lambda + \nu(E) = \int_E 1 d\lambda = \lambda(E) = \mu(E) + \nu(E)$$

Thus $\int_E (1-f) d\lambda = \mu(E)$ for any $\nu(E) < \infty$. This result extends to all $E \in \mathcal{M}$ since ν is σ-finite. Thus $d\mu/d\lambda = (1-f)$. Therefore

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda}\frac{d\lambda}{d\nu} = \frac{f}{1-f}$$

□

Exercise 17 Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $\mathcal{N}$ a sub-σ-algebra of $\mathcal{M}$, and $\nu = \mu|\mathcal{N}$. If $f \in L^1(\mu)$, there exists $g \in L^1(\nu)$ such that $\int_E f d\mu = \int_E g d\nu$ for all $E \in \mathcal{N}$; if g' is another such function then $g = g'$ ν-a.e.

Proof. Define λ on $\mathcal{N}$ by $\lambda(E) = \int_E f d\mu$, since $\rho \ll \nu$, the rest is obvious by the Radon-Nikodym theorem. □

Exercise 18 Prove Proposition 3.13c.

Proof. The second part is obvious:

$$\left|\int f d\nu\right| = \left|\int f \frac{d\nu}{d|\nu|} d|\nu|\right| \leq \int |f|\, d|\nu|$$

Suppose $d\nu = g d\mu$. Since

$$\int |f| d\nu = \int |f| g d\mu \leq \int |fg| d\mu = \int |f| d|\nu|$$

$L^1(|\nu|) \subset L^1(\nu)$. Conversely, suppose $f \in L^1(\nu)$, let $\nu = \nu_r + \nu_i$ where ν_r and ν_i are real and imaginaty part of ν, then $f \in L^1(|\nu_r| + |\nu_i|)$. Therefore

$$\int |f| d|\nu| \leq \int |f|(d|\nu_r| + d|\nu_i|) \leq \infty$$

which concludes $L^1(|\nu|) \subset L^1(\nu)$. □

Exercise 19 If ν, μ are complex measures and λ is a positive measure, then $\nu \perp \mu$ iff $|\nu| \perp |\mu|$ and $\nu \ll \lambda$ iff $|\nu| \ll \lambda$.

Proof. Suppose $\rho = |\nu_r| + |\nu_i| + |\mu_r| + |\mu_i|$, and $d\nu = f_1 d\rho$, $d\mu = f_2 d\rho$. If $\nu \perp \mu$, suppose the corresponding null sets be P and N. Obviously ν is null on P iff $|\nu|$ is null on P, similarly for μ. For the second part, suppose $\lambda(E) = 0$, if $\nu(E) = 0$, then for any $A \subset E$ that is measurable,

$$\lambda(A) = 0 = \int_A f_1 d\rho$$

therefore

$$\int_E |f_1| d\rho = |\nu|(E) = 0$$

which completes the proof. □

Exercise 20 If ν is a complex measure on $(X, \mathcal{M})$ and $\nu(X) = |\nu|(X)$, then $\nu = |\nu|$.

Proof. Suppose $d\nu = f d\mu$. Then for any measurable set $E \subset X$,

$$\nu(E) + \nu(E^c) = |\nu|(E) + |\nu|(E^c)$$

therefore

$$\int_E f d\mu + \int_{E^c} f d\mu = \int_E |f| d\mu + \int_{E^c} |f| d\mu, \quad \int_E (f - |f|) d\mu = \int_{E^c} (|f| - f) d\mu$$

Let $f = f_r + i f_i$, where f_r and f_i are real functions. Then

$$\int_E (f_r - |f|) d\mu + i \int_E f_i d\mu = \int_{E^c} (|f| - f_r) d\mu - i \int_{E^c} f_i d\mu$$

by comparing the real part and since $f_r \leq |f|$,

$$\int_E (f_r - |f|) d\mu = \int_{E^c} (|f| - f_r) d\mu = 0$$

therefore $|f| = f_r$ μ-a.e. Thus

$$|\nu|(E) = \int_E |f| d\mu = \int_E f d\mu = \nu(E)$$

□

Exercise 21 Let ν be a complex measure on $(X, \mathcal{M})$. If $E \in \mathcal{M}$, define

$$\mu_1(E) = \sup\left\{ \sum_1^n |\nu(E_j)| : n \in \mathbb{N},\ E_1, \cdots, E_n \text{ disjoint, } E = \bigcup_1^n E_j \right\},$$

$$\mu_2(E) = \sup\left\{ \sum_1^\infty |\nu(E_j)| : n \in \mathbb{N},\ E_1, \cdots \text{ disjoint, } E = \bigcup_1^\infty E_j \right\},$$

$$\mu_3(E) = \sup\left\{ \left| \int_E f d\mu \right| : |f| \leq 1 \right\}.$$

Then $\mu_1 = \mu_2 = \mu_3 = |\nu|$.

Proof. It is obvious that $\mu_1 \leq \mu_2$. To see that $\mu_2 \leq \mu_3$, let

$$f = \sum_{k=1}^\infty \frac{\overline{\nu(E_k)}}{|\nu(E_k)|} \chi_{E_k}$$

obviously $|f| \leq 1$. Suppose $a \in \{r, i\}$ and $b \in \{+, -\}$ then $1 \in L^1(\nu_a^b)$. Therefore by DCT,

$$\int f d\nu_a^b = \lim_{n \to \infty} \int \sum_{k=1}^n \frac{\overline{\nu(E_k)}}{|\nu(E_k)|} \chi_{E_k} d\nu_a^b = \sum_{k=1}^\infty \frac{\overline{\nu(E_k)}}{|\nu(E_k)|} \nu_a^b(E_k)$$

It follows that

$$\int f d\nu = \int f d\nu_r + i \int f d\nu_i = \int f d\nu_r^+ - \int f d\nu_r^- + i \int f d\nu_i^+ - i \int f d\nu_i^-$$
$$= \sum_{k=1}^{\infty} \frac{\overline{\nu(E_k)}}{|\nu(E_k)|} \nu(E_k) = \sum_{k=1}^{\infty} |\nu(E_k)|$$

Now show that $\mu_3 = |\nu|$. Let $f = \overline{d\nu/d|\nu|}$. By prop 3.13,

$$\int_E f d\nu = \int_E f \overline{f} d|\nu| = |\nu|(E) \leq \mu_3(E) \leq \left| \int_E 1 d\nu \right| = |\nu(E)| \leq |\nu|(E)$$

It remains to show that $\mu_3(E) \leq \mu_1(E)$. Suppose $|f| \leq 1$, then there exists a increasing sequence ϕ_k of simple functions which converges pointwise to f. Let

$$\phi_k = \sum_{j=1}^{n_k} c_{kj} \chi_{E_{kj}}$$

be the standard representation of ϕ_k. By DCT,

$$\int_E f d\nu_a^b = \lim_{k\to\infty} \sum_{j=1}^{n_k} c_{kj} \nu_a^b (E_{kj} \cap E)$$

hence

$$\int_E f d\nu = \lim_{k\to\infty} \sum_{j=1}^{n_k} c_{kj} \nu(E_{kj} \cap E)$$

thus

$$\left| \int_E f d\nu \right| = \lim_{k\to\infty} \left| \sum_{j=1}^{n_k} c_{kj} \nu(E_{kj} \cap E) \right| \leq \lim_{k\to\infty} \sum_{j=1}^{n_k} |\nu(E_{kj} \cap E)| \leq \mu_1(E)$$

□

Exercise 22 If $f \in L^1(\mathbb{R}^n)$, $f \neq 0$, there exist $C, R > 0$ such that $Hf(x) \geq C|x|^{-n}$ for $|x| > R$.

Proof. If $||f|| > 0$, then there exists $R \in (0, \infty)$ such that $\int_{B_R(0)} |f| dm > 0$. If $x \in \mathbb{R}^n \backslash B_R(0)$, then $B_R(0) \subset B_{2|x|}(x)$. Therefore

$$Hf(x) \geq A_{2|x|}|f|(x) \geq \frac{1}{|x|^n m(B_2(0))} \int_{B_R(0)} |f| dm = C|x|^{-n}$$

If $\alpha \in (0, C/2R^n)$ and $R < |x| < (C/\alpha)^{1/n}$, then $Hf(x) \geq \alpha$ and

$$m(\{x : Hf(x) > \alpha\}) \geq m(B_{(C/\alpha)^{1/n}}) - m(B_R(0)) > Cm(B_1(0))/2\alpha$$

□

Exercise 23 A useful variant of the Hardy-Littlewood maximal function is

$$H^* f(x) = \sup \left\{ \frac{1}{m(B)} \int_B |f(y)| dy : B \text{ is a ball and } x \in B \right\}$$

Show that $Hf \leq H^* f \leq 2^n Hf$.

Proof. It is clear that $Hf(x) \leq H^* f(x)$. For the other inequality, suppose $x \in B_r(y)$, then $B_r(y) \subset B_{2r}(x)$, by writing down definitions it is easy to see the inequality is true. □

Exercise 24 If $f \in L^1_{\text{loc}}$ and f is continuous at x, then x is in the Lebesgue set of f.

Proof. Since f is continuous at x, $\forall \epsilon > 0$, there exists $r > 0$ such that $|f(y) - f(x)| < \epsilon$ for any $y \in B_r(x)$. Therefore

$$\frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)| dy \leq \frac{1}{m(B(r,x))} \int_{B(r,x)} \epsilon dy = \epsilon$$

therefore $\lim_{r \to 0} A_r |f(y) - f(x)| = 0$. □

Exercise 25 If E is a Borel set in $\mathbb{R}^n$, the density $D_E(x)$ of E at x is defined as

$$D_E(x) = \lim_{r \to 0} \frac{m(E \cap B(r,x))}{m(B(r,x))}$$

whenever the limit exists.

(a) Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.

(b) Find examples of E and x such that $D_E(x)$ is a given number $\alpha \in (0,1)$ or such that $D_E(x)$ does not exist.

Proof. (a) Define $\mu(A) = m(E \cap A)$. Then $\mu = \chi_E dm$. Suppose $m(A) < \infty$, for any $\epsilon > 0$, there exists an open set U such that $m(U) < m(A) + \epsilon$. Therefore $\mu(U) < \mu(A) + \epsilon$. Now for any A that is measurable, take A_k such that $m(A_k) < \infty$ and $A = \cup_k A_k$. Then for each k there exists an open set U_k such that $\mu(U_k) < \mu(A_k) + 2^{-k}\epsilon$. Thus $\mu(\cup_k U_k \backslash A) < \epsilon$, which implies that μ is regular. So for a.e. $x \in \mathbb{R}^n$,

$$D_E(x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{m(B_r(x))} = \chi_E(x)$$

(b) Suppose $E = \{x : x_i > 0\}$ and $x = 0$. Then $D_E(x) = 2^{-n}$. □

Exercise 26 If λ and μ are positive, mutually singular Borel measures on $\mathbb{R}^n$ and $\lambda + \mu$ is regular, then so are λ and μ.

Proof. Condition (i) holds trivially. For condition (ii), since $\mu \perp \lambda$, suppose P is μ-null and P^c is λ-null. Since $\lambda + \mu$ is regular, for any $E \subset P$ that is Borel measurable, there exists an open set U such that $\lambda(U) < \lambda(U) + \mu(U) < \lambda(E) + \epsilon$ for any $\epsilon > 0$, therefore λ is regular. The same goes for μ. □

Exercise 27 Verify Example 3.25.

Proof. (a) is obvious since $\lim_{x \to \infty} T_F(x) = F(\infty) - F(-\infty) < \infty$.

(b) If $F, G \in BV$, then

$$\sum_1^n |aF(x_j) + bG(x_j) - aF(x_{j-1}) - bG(x_{j-1})| \leq \sum_1^n (a|F(x_j) - F(x_{j-1})| + b|F(x_j) - F(x_{j-1})|)$$

therefore $T_{aF+bG}(x) \leq aT_F(x) + bT_G(x)$, the rest is obvious.

(c) Since F is differentiable on $\mathbb{R}$ and F' is bounded, by the mean value theorem,

$$\sum_1^n |F(x_j) - F(x_{j-1})| = \sum_n^1 |F'(c_j)||x_j - x_{j-1}| \leq |M||b - a|$$

therefore the variation on $[a,b]$ is bounded.

(d) $\sin x \in BV([a,b])$ by (c). To see $\sin x \notin BV$, just take $x_j = -\pi/2 + 2\pi j$.

(e) Just take $x_j = 1/(\pi/2 + 2\pi j)$. □

Exercise 30 Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is $\mathbb{Q}$.

Proof. Enumerate rational numbers $\mathbb{Q} = \{q_i\}$. Define $f := \sum_i 2^{-i}\chi_{(q_i,\infty)}$. Obviously $f(x) \leq f(y)$ if $x \leq y$. Suppose $\epsilon > 0$ and take N such that $2^{-N} < \epsilon$. There exists $\delta > 0$ such that $(x-\delta, x) \cup (x+\delta)$ does not contain $q_1, \cdots, q_N$. If $y \in (x-\delta, x)$, then

$$f(x) \geq f(y) = f(x) - \sum_{y \leq q_i < x} 2^{-i} \geq f(x) - \epsilon$$

therefore f is left continuous. If $y \in (x, x+\delta)$ then

$$f(x) \leq f(y) = f(x) + \sum_{x \leq q_i < y} 2^{-i}$$

if $x \notin \mathbb{Q}$, by the same argument f is right continuous. If $x \in \mathbb{Q}$, say $x = q_n$, then

$$f(x) + 2^{-n} \leq f(y) < f(x) + 2^n + \epsilon$$

which is not continuous. □