NOTES ON DIFFERENTIAL TOPOLOGY

NAN AN

Contents

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\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker d^{(k)}_{U\cup V} \arrow[r, "\alpha_1^{(k)}"] \arrow[d] & \ker(d^{(k)}_U \oplus d^{(k)}_V) \arrow[r, "\beta_1^{(k)}"] \arrow[d] & \ker d^{(k)}_{U\cap V} \arrow[d] \arrow[dddll, "\delta"] & \\
0 \arrow[r] & \Omega^k(U\cup V) \arrow[r, "\alpha^{(k)}"] \arrow[d] & \Omega^k(U)\oplus\Omega^k(V) \arrow[r, "\beta^{(k)}"] \arrow[d] & \Omega^k(U\cap V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^{k+1}(U\cup V) \arrow[r, "\alpha^{(k+1)}"] \arrow[d] & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) \arrow[r, "\beta^{(k+1)}"] \arrow[d] & \Omega^{k+1}(U\cap V) \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker} d^{(k)}_{U\cup V} \arrow[r, "\alpha_2^{(k+1)}"] \arrow[d] & \operatorname{coker}(d^{(k)}_U \oplus d^{(k)}_V) \arrow[r, "\beta_2^{(k+1)}"] \arrow[d] & \operatorname{coker} d^{(k)}_{U\cap V} \arrow[r] \arrow[d] & 0 \\
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\end{tikzcd}
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1. Manifolds and Submanifolds

Definition 1.1. Let X be a topological space. For $n \in \mathbb{N}^+$, X is called a *n-dimensional manifold* if for every $x \in X$, there exists $U \subseteq X$ open with $x \in U$ and $\varphi : U \to \mathbb{R}^n$ such that φ is a homeomorphism between U and $\varphi(U)$, and $\varphi(U)$ is open in $\mathbb{R}^n$. Such (φ, U) is called a *chart*.

Example 1.2. (a) $X = \{(x, |x|) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$. $(\mathbb{R}, \varphi)$ where $\varphi : X \to \mathbb{R}, (x, |x|) \mapsto x$ is a chart covers X since φ is continuous and the inverse $z \mapsto (z, |z|)$ is also continuous. Therefore X is a 1-dimensional manifold.

(b) Suppose $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and take $(x_0, y_0) \in S^1$. If $|x_0| \neq 1$, then $\{(x, y) \in S^1 \mid yy_0 > 0\}$ and $\varphi : (x, y) \mapsto x$ gives a chart, since $\operatorname{im}(\varphi) = (-1, 1)$ and its inverse $x \mapsto (x, \operatorname{sgn}(y_0)\sqrt{1 - x_0^2})$ are continuous. If $|y_0| \neq 1$, one can construct a chart by simply exchanging x_0 and y_0 in the $|x_0| \neq 1$ case.

Definition 1.3. Suppose $U \subset\subset \mathbb{R}^n$, for a positive integer r, a function

$$f : U \to \mathbb{R}^m, (x_1, \cdots, x_n) \mapsto (f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n))$$

is said to be *C^r-differentiable*, if for each $f_i(x_1, \cdots, x_n)$, all partial derivatives

$$\frac{\partial^\alpha f_i}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} : U \to \mathbb{R}$$

exists and continuous for every $\alpha_1, \cdots, \alpha_n \in \mathbb{N}^+$ such that $\alpha = \alpha_1 + \cdots + \alpha_n \leq r$ and each point in U. Also, f is *C^0-differentiable* if it is continuous, and *C^∞-differentiable* (or *smooth*) if it is C^r-differentiable for any non-negative integer r. For a smooth function f, if it satisfies the Cauthy-Riemann equation for each complex variable, then it is *C^w-differentiable* (or *analytic*).

Definition 1.4. Let X be an n-dimensional manifold, two charts (φ, U) and (ψ, V) are said to have a *C^r-overlap* if $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \psi(U \cap V)$ is a C^r-diffeomorphism.

Definition 1.5. A family of charts $\{(q_i, U_i)\}_{i \in I}$ of X is called an *atlas* if $X = \bigcup_{i \in I} U_i$. An atlas of X is called a *C^r-atlas* if all overlaps are C^r. A maximal C^r-atlas α (with respect to inclusion) is called a *C^r-differentiable structure* and (X, α) is called a *C^r-mainfold*. A C^r-manifold with $r \geq 1$ is called a *smooth manifold*.

Proposition 1.6. *Let M be a C^r-manifold and Φ is a C^r-atlas on M. Then there is a unique maximal C^r-atlas on M which contains Φ.*

Proof. Let T be the collection of all the charts of all the C^r-atlases that is compactable with Φ (i.e. for any chart in these atlases, it has a C^r-overlap with with all the charts in Φ). T is an atlas since all the charts have C^r-overlaps. T is maximal, since if $T \subseteq T'$, T' is compactable with T and therefore contained in T. $\square$

Example 1.7. (a) Suppose $D \subset\subset \mathbb{R}^n$, let $f : D \to \mathbb{R}^m$ be a continuous map, define $\mathit{\Gamma}_f = \{(x, f(x)) \mid x \in D\}$. The chart $P_1 : \mathit{\Gamma}_f \to D, (x, f(x)) \mapsto x$ gives an analytic differentiable structure.

(b) $S^n = \{\underline{x} \in \mathbb{R}^{n+1} \mid |\underline{x}|_2 = 1\}$. The following charts form a C^∞-atlas on S^n. For $i \in \{\pm 1\}$ and $i \in \{1, \cdots, n+1\}$, define $(q_{i,s}, U_{i,s})$ where $U_{i,s} = \{\underline{x} \in S^n \mid sx_i > 0\}$ and $q_{i,s}(x_1, \cdots x_{n+1}) = (x_1, \cdots, \hat{x_i}, \cdots, x_{n+1})$. Take (i, s) and (j, t) such that $i < j$, then the overlap

$$(q_{i,s} \circ p_{j,t}^{-1})(\underline{z}) = q_{i,s}(z_1, \cdots, \hat{z_j}, \cdots, t\sqrt{1 - |\underline{z}|_2}, \cdots, z_n)$$

is smooth.

Definition 1.8. Let M be a n-dimensional manifold with a C^r-differentiable structure α_r, and let N be a non-empty subset of M. N is called a *k-dimensional submanifold* of M if for all $x \in N$, $\exists (\varphi, U) \in \alpha_r$ such that $N \cap U = \varphi^{-1}(\mathbb{R}^k \times \{0\})$ where $0 \in \mathbb{R}^{n-k}$.

Remark 1.9. A chart of M that satisfies the definition above is called a *submanifold chart* for N. If N is a submanifold of M, then the set $\{(\varphi|_{U\cap N}, U\cap N)|(\varphi,U)\in\alpha_r\}$ is a C^r-atlas of N.

Example 1.10. (a) $N=\{(x,y,z)\in\mathbb{R}^3 \mid z=x^2+y^2\}$ is a submanifold of $\mathbb{R}^3$, because $(\varphi,\mathbb{R}^3)$ defined via $\varphi(x,y,z)=(x,y,z-x^2-y^2)$ is a submanifold chart, since (1) $\varphi:\mathbb{R}^3\to\mathbb{R}^3$ is a homeomorphism; (2) $(\varphi,\mathbb{R}^3)$ has a C^∞-overlap with $\mathrm{id}_{\mathrm{id},\mathbb{R}^3}$; (3) $\varphi^{-1}(R^2\times\{0\})=N$.

(b) $N=\{(x,|x|)\mid x\in\mathbb{R}\}$ is not a C^1-submanifold of $\mathbb{R}^2$.

Remark 1.11. (a) If M is a smooth manifold and let (φ,U) and (ψ,V) be two charts with C^1-overlap, and $U\cap V\neq\varnothing$, then the dimension of the target spaces are the same because the transition map must be C^1-diffeomorphism.

(b) Let M be a n-dimensional manifold and $\psi:V\to\mathbb{R}^m$ an open embedding, then $m=n$. Therefore the dimension of a manifold relies only on its topological structure.

Definition 1.12. A topological T^2 space $(X,\mathcal{I})$ is called *paracompact* if for every open cover $\mathcal{C}\subseteq\mathcal{I}$ of X, there is a refinement $\mathcal{S}\subseteq\mathcal{I}$ (i.e. $\mathcal{S}$ is an open cover of X such that for any $U\in\mathcal{S}$, $\exists V\in\mathcal{C}$ with $U\subseteq V$) which is locally finite (i.e. $\forall x\in X,\exists K\subset X$ an open neighbourhood of x such that only finitely elements of $\mathcal{S}$ intersect K).

Theorem 1.13 (Smirnov metrization theorem[1]). *Let X be a T^2-space, these statements are equivalent:*

(a) X is metrizable;

(b) X is paracompact and locally metrizable, i.e. $\forall x\in X,\exists U\in\mathcal{I}$ with $x\in U$ such that $(U,\mathcal{I}_U)$ is metrizable.

Remark 1.14. This lecture will only consider manifolds that are T_2, second countable, paracompact, and at most countablly many connected components. For those manifolds, the following tatements are true[2]: (a) every smooth manifold has a unique C^∞-differential structure, i.e. $\exists!\alpha_\infty$ such that $\alpha_\infty\cap\alpha\neq\varnothing$; (b) there exists C^0-manifolds that do not exist a C^1-differential structure on it; (c) such manifolds are always metrizable.

Definition 1.15. Let (M,α_r) and (N,β_r) be smooth manifolds and $f:M\to N$ a map between them. f is said to be *differentiable* at $x_0\in M$ if there exists $(\varphi,U)\in\alpha_r$ with $x_0\in U$ and $(\psi,V)\in\beta_r$ with $f(U)\subseteq V$ such that $\psi\circ f\circ\varphi^{-1}:\varphi(U)\to\psi(V)$ is differentiable at $\varphi(x_0)$.If $\psi\circ f\circ\varphi^{-1}$ is C^{r-1}-differentiable and $\psi\circ f\circ\varphi^{-1}$ is differentiable at $\varphi(x_0)$, then f is *r-times differentiable* at x_0. f is *C^r-differentiable* if $\psi\circ f\circ\varphi^{-1}$ is C^r-differentiable.

Example 1.16. (a) For $f:\mathbb{R}^n\to\mathbb{R}^m$, the definition of C^r-differentiability is consistent with the previous one.

(b) Every C^r-map $f:S^n\to M$ is a restriction of a C^r-map $\mathbb{R}^{n+1}\backslash\{0\}\to M$ given the construction $F(\underline{x})=f(\underline{x}/|\underline{x}|_2)$

Definition 1.17. Let M be a C^1-manifold, $T=\{(x,\varphi,U,v):x\in U\subset M,(\varphi,U)\in\alpha_1,v\in\mathbb{R}^n\}$. Define $x\sim y$: $x=y$ and $D(\psi\circ\varphi^{-1})(\varphi(x))v=w$. The equivalent classes in $T/\sim$ is called a *tangent vector* of M at x. $T/\sim$ is called the *tangent bundle* of M. $T_xM=\{[x,\varphi,U,v]\mid x\in U\}$ is called the *tangent space* of M at x. The tangent space obviously admits a linear structure. For $f\in C^1(M,N)$,

(a) it defines a linear map $Tf:TM\to TN$ via $Tf([x,\varphi,U,v])=[f(x),\psi,V,D(\psi\circ f\circ\varphi^{-1})(\varphi(x))v]$, called the *derivative* of f;

(b) the restriction of Tf to T_xM, denoted by T_pf, is called the *derivative of f at x*.

[1]This theorem is not proved here.

[2]The statements listed in this remark are not proved in the lecture, I just take the word from the lecturer.

Example 1.18. (a) The inclusion $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$ is analytic with respect to the differential sturcture given in Example 1.7 (one need to specify this as there are differential structures on S^7 which are not compactable with it[3]). Let $1 \leq i \leq n+1$, $\varepsilon = \{\pm 1\}$, then

$$(\mathrm{id}_{\mathbb{R}^{n+1}} \circ \iota \circ \varphi_{i,\varepsilon}^{-1})(\underline{z}) = (z_1, \cdots, z_{i-1}, \sqrt{1 - |\underline{z}|_2^2}\varepsilon, z_i, \cdots, z_n)$$

therefore $\mathrm{id}_{\mathbb{R}^{n+1}} \circ \iota \circ \varphi_{i,\varepsilon}^{-1} \in C^w(B_1(\underline{0}), \mathbb{R}^{n+1})$, $\iota \in C^w(S^n, \mathbb{R}^{n+1})$.

(b) For S^1, take $(\varphi_{2,1}, U_{2,1})$ around $p = (x_0, y_0)$. Then $T_pS^1 = \{[p, \varphi_{2,1}, U_{2,1}, w] \mid w \in \mathbb{R}\}$. To see the intuition picture, compute $T_p\iota : T_pS^1 \to T_{\iota(p)\mathbb{R}^2}$,

$$T_p\iota([p, \varphi_{2,1}, U_{2,1}, v]) = [p, \mathrm{id}_{\mathbb{R}^2}, D(\mathrm{id}_{\mathbb{R}^2} \circ \iota \circ \varphi^{-1})(\varphi_{2,1}(p))v]$$

with

$$D(\mathrm{id}_{\mathbb{R}^2} \circ \iota \circ \varphi^{-1})(x_0)(v) = D(\begin{bmatrix} x \\ \sqrt{1-x^2} \end{bmatrix})(x_0)v = \begin{bmatrix} 1 \\ -\frac{x_0}{\sqrt{1-x_0^2}} \end{bmatrix} v$$

which corresponds to the tangent line of S^1.

Remark 1.19. $N \subseteq M$ a smooth submanifold of M, identify T_pN with $T_p\iota(T_pN) \subseteq T_p(M)$, since there is a canonical isomorphism.

Definition 1.20. Suppose $f \in C^r(M, N)$, a point $p \in f(M)$ is called a *regular value* if for any $x \in f^{-1}(p)$, the derivative of f at x is surjective.

Proposition 1.21. *Let $f : U \to \mathbb{R}^k$ be a C^r-map from an open set in $\mathbb{R}^n$, b a regular value and $M = f^{-1}(b)$. Then M is a C^r-submanifold of $\mathbb{R}^n$.*

Proof. (a) For any $a \in M$, the jacobian matrix of f at a is of rank r since the differential is surjective. Consider the map $F : U \to \mathbb{R}^n, \underline{x} \mapsto (f(x), x_{k+1}, \cdots, x_n)$, its jacobian $JF(a)$ is of rank n. By the inverse function theorem, there exists $V_1 \subset\subset \mathbb{R}^k, V_2 \subset\subset \mathbb{R}^{n-k}$ and a C^r-inverse $G : V_1 \times V_2 \to W$ with $b \in V_1$, $(a_{r+1}, \cdots, a_n) \in V_2$ and $W = G(V_1 \times V_2)$ defined on a open neighbourhood of $(b, a_{k+1}, \cdots, a_n)$ such that

$$\underline{x} = (F \circ G)(\underline{x}) = (f(G(\underline{x})), G(\underline{x})_{k+1}, \cdots, G(\underline{x})_n),$$

which implies

$$(f \circ G)(x_1, \cdots x_n) = (x_1, \cdots, x_k).$$

therefore $g = G(b, \cdot) : V_2 \to W \cap M$ is a bijective C^1-map. Denote $h : W \cap M \to \{0\} \times V_2, x \mapsto (0, g^{-1}(x))$, then $W \cap M = h^{-1}(\{b\} \times \mathbb{R}^{n-k})$. Hence M is a $n-k$ dimensional submanifold. From another point view, $h : W \cap M \to \mathbb{R}^{n-r}$ is a chart for the manifold M. □

Theorem 1.22 (Regular value theorem)**.** *Let $f \in C^r(M, N)$, $r \geq 1$, and $q \in f(M)$ be a regular value of f, then $f^{-1}(q)$ is a C^r-submanifold of M of dimension* $\dim M - \dim N$.

Proof. For all $x \in f^{-1}(q)$, take a chart (φ, U) around x such that $f \circ \varphi^{-1}(U)$ is contained in one chart (ψ, V) in N. Suppose $g : \varphi^{-1}(U) \to \mathbb{R}^n, \psi \circ f \circ \varphi^{-1}$, since p is a regular value of g, by proposition 1.21, there exists (φ', U') such that $\varphi'^{-1}(\{0\} \times \mathbb{R}^{m-n}) = U' \cap g^{-1}(p) \cong \varphi(U') \cap f^{-1}(p)$, thus $f^{-1}(q)$ is a submanifold of M. □

Example 1.23. Suppose $f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto y^2 - x^3$. The derivative of $f^{-1}(0)$ is zero only at $(0, 0)$ and $f^{-1}(0)$ is not a C^1-manifold.

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[3]Found by John Milnor in 1956.

Example 1.24. For $0 < r_1 < r_2$, define a *torus*:

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - r_2)^2 + z^2 = r_1^2\}$$

then M is a C^1-manifold of $\mathbb{R}^3$, and M is C^1-diffeomorphic to $S^1 \times S^1$.

Proof. Apply proposition 1.21 here. Let $F(x, y, z) = (\sqrt{x^2 + y^2} - r_2)^2 + z^2 - r_1^2$, then $M = F^{-1}(0)$. Take $(x_0, y_0, z_0) \in M$, claim that $DF(x_0, y_0, z_0) \neq (0, 0, 0)$.

If $z_0 \neq 0$, then $F_z(x_0, y_0, z_0) = 2z_0 \neq 0$. If $z_0 = 0$ and $x_0 \neq 0$, then

$$\left|\frac{\partial F}{\partial x}(x_0, y_0, z_0)\right| = 2(\sqrt{x_0^2 + y_0^2} - r_2^2)\frac{|x_0|}{\sqrt{x_0^2 + y_0^2}} = \frac{|r_1 x_0|}{\sqrt{x_0^2 + y_0^2}} \neq 0$$

If $z_0 = 0$ and $x_0 = 0$ but $y_0 \neq 0$, then

$$\left|\frac{\partial F}{\partial y}(0, y_0, 0)\right| = |r_1| \neq 0$$

therefore M is indeed a submanifold. For the second part, suppose a map

$$P : S^1 \times S^1 \to M, (\alpha, \theta) \mapsto (r_2 \cos\theta + r_1 \cos\alpha \cos\theta, r_2 \sin\theta + r_1 \cos\alpha \sin\theta, r_1 \sin\alpha)$$

where $\alpha, \theta \in [0, 2\pi)$. P is smooth, and its inverse can be given easily and also smooth, therefore P is a diffeomorphism. $\square$

Definition 1.25. Suppose (M, α_∞) a C^∞-manifold, and $p \in M$. Define the *stalk* at p (of the sheaf of all smooth functions of M)

$$C^\infty(M, \mathbb{R})_p = \{(f, U) \mid f \in C^\infty(U, \mathbb{R}), U \subset\subset M, p \in U\}/\sim$$

with $(f, U) \sim (g, V)$ iff there exists open set W such that $f|_W = g|_W$. The equivalence class $[f, U]_p$ in $C^\infty(M, \mathbb{R})_p$ is called the *germ* of (f, U) and p. An $\mathbb{R}$-linear map $\partial : C^\infty(M, \mathbb{R})_p \to \mathbb{R}$ that satisfies the Leibniz rule $\partial([fg]_p) = f(p)\partial([g]_p) + g(p)\partial([f]_p)$ is called a *derivative*.

Example 1.26. An example of derivatives:

$$\partial_{v,p} : C^\infty(M, \mathbb{R})_p \to \mathbb{R}, [f]_p \mapsto \frac{d(f \circ \varphi^{-1}(\varphi(p) + tv))}{dt}\Big|_{t=0}$$

where $f \circ \varphi^{-1}(\varphi(p) + tv)$ is seen as a function from $\mathbb{R}$ to $\mathbb{R}$ when differentiating.

Definition 1.27. Let (M, α_r) be a smooth manifold. Take $p \in M$, and I an open interval.

(a) A C^r-curve $c : I \to M$ is said to *start at* p if $0 \in I$ and $c(0) = p$.

(b) Two C^r-curves $c : I \to M$ and $d : J \to M$ with $0 \in I \cap J$ are called *jet-equivalent* if $c(0) = d(0)$ and $\exists(\varphi, U) \in \alpha_r$ around $c(0)$ such that $(\varphi \circ c)^{(i)}(0) = (\varphi \circ d)^{(i)}(0)$ for $1 \leq i \leq r$.

Remark 1.28. (a) Let (M, α_n) be a C^1-manifold and $c : I \to M$ and $d : J \to M$ C^1-curves starting at $p_0 \in M$, then $T_0 c = T_0 d$ iff they are jet-equivalent, since $T_0 c([0, \mathrm{id}_I, I, 1]) = [p, \varphi, U, (\varphi \circ c)'(0)]$.

(b) The set $\mathrm{Der}_p(M) = \{A : C^\infty(M, \mathbb{R})_p \to \mathbb{R} \mid A \text{ is a derivative}\}$ is called *the set of derivatives at* p.

Lemma 1.29. *Let M^n be a C^∞-manifold and (φ, U) be a C^∞-chart of M and $p \in U$. Suppose that $\varphi(p) = \underline{0} \in \mathbb{R}^n$. Let $\varphi_i : U \to \mathbb{R}$ be the i-th coordinate of φ, i.e. $\varphi = (\varphi_1, \cdots, \varphi_n)$, then the map*

$$\Phi : \mathrm{Der}_p(M) \to \mathbb{R}^n, A \mapsto (A(\varphi_1), \cdots, A(\varphi_n))$$

is an $\mathbb{R}$-linear isomorphism.

Proof. The map is $\mathbb{R}$-linear via construction. To see it is surjective, take $v \in \mathbb{R}^n$, define $A_v \in \mathrm{Der}_p(M)$ via $A_v(f_p) = df(\varphi^{-1}(tv))/dt|_{t=0}$, then $A(\varphi_i) = d(\pi_i \circ \varphi \circ \varphi^{-1}(tv))/dt = v_i$, where π_i is the coordinate projection. Therefore $\Phi(A_v) = v$.

Now verify its injectivity. Claim that $\ker(\Phi) = \{0\}$. Suppose $A \in \ker(\Phi)$, then $A(\varphi_1) = \cdots = A(\varphi_n) = 0$. Take $f_p \in C^\infty(M, \mathbb{R})_p$ and point q around p, then

$$f(q) - f(p) = (f \circ \varphi^{-1})(\varphi_1(q), \cdots, \varphi_n(q)) - (f \circ \varphi^{-1})(0)$$

by the fundamental theorem of calculus,

$$\begin{aligned} f(q) - f(p) &= \int_0^1 \frac{d(f \circ \varphi^{-1})(t\varphi_1(q), \cdots, t\varphi_n(q))}{dt} dt \\ &= \sum_{j=1}^n \varphi_j(q) \int_0^1 \frac{d(f \circ \varphi^{-1})}{\partial x_j}(t\varphi(q)) dt \\ &= \sum_{j=1}^n \varphi_j(q) g_j(q), \quad g_j(q) = \int_0^1 \frac{d(f \circ \varphi^{-1})}{\partial x_j}(t\varphi(q)) dt \end{aligned}$$

thus

$$A(f_p) = \sum_{j=1}^n A(\varphi_j g_j) = \sum_{j=1}^n (\varphi_j(p) A(g_j) + g_j(p) A(\varphi_j)) = 0$$

therefore $D = 0$. Note that the inverse of Φ, $\Psi : \mathbb{R}^n \to \mathrm{Der}_p M$ is given by

$$v \mapsto \frac{d(- \circ \varphi^{-1}(tv))}{dt}\Big|_{t=0}$$

it can be easily verified that $\Phi \circ \Psi = \mathrm{id}$ and $\Psi \circ \Phi = \mathrm{id}$. □

Proposition 1.30. *Let M be C^1-manifold and $p \in M$, and let the collection of jet-equivalence classes of C^1-curves starting at P be $C(M)$. Then $T_pM \cong C(M)$ naturally, i.e. for every map $f \in C^1(M, N)$, diagram*

$$\begin{array}{ccc} T_pM & \xrightarrow{\sim} & C(M) \\ \downarrow{\scriptstyle Tf} & & \downarrow{\scriptstyle C_f} \\ T_{f(p)}N & \xrightarrow{\sim} & C(N) \end{array}$$

commutes, where $C_f : C(M) \to C(N), [c] \mapsto [f \circ c]$.

Proof. The isomorphism is $g : T_pM \to C, [p, \varphi, U, v] \mapsto [c]$ with $c(t) = \varphi^{-1}(\varphi(p) + tv)$, and its inverse is given by $h : [c] \mapsto [p, \varphi, U, (\varphi \circ c)'(0)]$. It is indeed an isomorphism, since

$$h \circ g([p, \varphi, U, v]) = [p, \varphi, U, \frac{d}{dt}(\varphi(p) + tv)] = [p, \varphi, U, v]$$

and

$$g \circ h([c]) = [\varphi^{-1}(\varphi(p)) + t(\varphi \circ c)'(0)] = [c]$$

and the diagram commutes since

$$\begin{aligned} g \circ Tf([p, \varphi, U, v]) &= [\psi^{-1}(\psi \circ f(p) + tD(\psi \circ f \circ \varphi^{-1})(p)v)] \\ &= [f \circ \varphi^{-1}(\varphi(p) + tv)] = C_f \circ g([p, \varphi, U, v]) \end{aligned}$$

□

Proposition 1.31. *Let M be a C^∞-manifold and $p \in M$. Then $T_pM \cong \mathrm{Der}_p(M)$ naturally, i.e. for every map $f \in C^\infty(M, N)$, diagram*

$$\begin{tikzcd} \mathrm{Der}_p M \arrow[r, "\sim"] \arrow[d, "Rf"] & T_pM \arrow[d, "Tf"] \\ \mathrm{Der}_{f(p)} N \arrow[r, "\sim"] & T_{f(p)}N \end{tikzcd}$$

commutes, where $Rf : A(-) \mapsto A(- \circ f)$

Proof. The isomorphism is $g : \mathrm{Der}_p(M) \to T_pM,\ A \mapsto [p, \varphi, U, \Phi(A)]$. g is an $\mathbb{R}$-linear homeomorphism by lemma 1.29. Since

$$A(\psi_i \circ f) = \frac{d(\psi_i \circ f \circ \varphi^{-1})(tv)}{dt}\Big|_{t=0}, \quad v = \begin{bmatrix} A(\varphi_1) \\ \cdots \\ A(\varphi_n) \end{bmatrix},$$

by the chain rule

$$(g \circ R_f(A))_i = A(\psi_i \circ f) = \sum_{j=1}^{n} A(\varphi_j) \frac{d(\psi_i \circ f \circ \varphi^{-1})}{dx_j} = (D(\psi \circ f \circ \varphi^{-1})(p)v)_i = (Tf \circ g(A))_i$$

therefore the diagram commutes. □

Remark 1.32. (a) The tangent vectors $[p, \varphi, e_1], \cdots, [p, \varphi, e_n]$ are denoted by $\frac{\partial}{\partial x_1}|_p, \cdots \frac{\partial}{\partial x_n}|_p$, emphasizing their description as derivatives.

(b) For a C^r-manifold M^n, define a C^{r-1}-differentiable structure on TM with charts $(\varphi, U) \in \alpha_r$, $T\varphi : TU \to \varphi(U) \times \mathbb{R}^n, [p, \varphi, U, v] \mapsto (\varphi(p), v)$. Obviously it is bijective and $\mathbb{R}$-linear, to see it is C^{r-1}, compute the overlaps: for $(\varphi, U), (\psi, V) \in \alpha_r$,

$$(T\varphi)(T\psi)^{-1}(\psi(p), w) = (\varphi(p), D(\varphi \circ \psi^{-1})(\psi(p))w)$$

the second component is of C^{r-1}.

Definition 1.33. Let (M, α_r) be a smooth manifold, its tangent bundle TM is said to be *trivial* if there exists a C^{r-1}-diffeomorphism $F : TM \to M \times \mathbb{R}^n$ such that diagram

$$\begin{tikzcd} TM \arrow[rr, "F"] \arrow[dr] & & M \times \mathbb{R}^n \arrow[dl] \\ & M & \end{tikzcd}$$

commutes, where the maps $TM \to M$ and $M \times \mathbb{R}^n \to M$ are canonical projections.

Example 1.34. (a) For S^1, a trivialization is given by $TS^1 \to S^1 \times \mathbb{R}, [p, \varphi, U, v] \mapsto (p, v)$.

(b) $M = \{P(\theta, s) \mid \theta \in (-\pi, \pi], s \in (-1, 1)\}$, where

$$P(\theta, s) = \left(2\cos\theta + s\cos\frac{\theta}{2}\cos\theta, 2\sin\theta + s\cos\frac{\theta}{2}\sin\theta, s\sin\frac{\theta}{2}\right)$$

then M is a mobius strip. Its tangent bundle is not trivial.

Proof. Consider the following C^∞-maps:

$$E_1(\theta) : \mathbb{R} \to TM, \theta \mapsto \left(P(\theta, 0), \frac{\partial P}{\partial \theta}(\theta, 0)\right)$$

$$E_2(\theta) : \mathbb{R} \to TM, \theta \mapsto \left(P(\theta, 0), \frac{\partial P}{\partial s}(\theta, 0)\right)$$

Assume there is a trivialization Φ, let $f_i = \pi \circ \Phi \circ E_i$, where $\pi : M \times \mathbb{R}^2 \to \mathbb{R}^2$ is the projection. Since for each $\theta \in \mathbb{R}$, $\operatorname{span}\{f_1(\theta), f_2(\theta)\} = \mathbb{R}^2$, therefore $\det(f_1(\theta), f_2(\theta)) \neq 0$. However, since $\det(f_1(0), f_2(0)) = -\det(f_1(2\pi), f_2(2\pi))$, by applying the intermediate value theorem there is a contradiction. $\square$

Definition 1.35. Let (M^n, α_r) be a smooth manifold, i.e. $r \geq 1$, a map $X \in C^{r-1}(M, TM)$ such that $X(p) \in T_pM$ for all $p \in M$ is called a *C^{r-1}-vector field.*

Proposition 1.36. *Let (M, α_r) be a smooth manifold, then TM is trivial iff there exist $X_1, \cdots, X_n$ (C^{r-1}-vector fields) on M such that for all p, $\operatorname{span}\{X_1(p), \cdots, X_n(p)\} = \mathbb{R}^n$.*

Proof. If there exists vector fields $X_1, \cdots, X_n$ that spans $\mathbb{R}^n$ for every point of M, suppose basis transformation $P : \{e_1, \cdots, e_n\} \to \{X_1(p), \cdots, X_n(p)\}$, then the map $\Phi : [p, \varphi, U, v] \mapsto (p, P^{-1}v)$ is a trivialization. To verify it is well defined, suppose two charts $(U, \varphi), (V, \psi)$ with $p \in U \cap V$, denote $T = D(\psi \circ \varphi^{-1})(p)$. Then for the basis transformation $P' : \{e_1, \cdots, e_n\} \to \{TX_1(p), \cdots, TX_n(p)\}$, one have $P' = TP$. Thus

$$\Phi([p, \psi, V, Tv]) = (p, (TP)^{-1}Tv) = (p, P^{-1}v)$$

On the other hand, if TM is trivializable, suppose a trivialization Φ, then just take $X_i(p) = \Phi(p, e_i)$. $\square$

Remark 1.37. The map E_2 in example 1.34 cannot be extended to a vector field over M because it cannot be defined on S^1 since $E_2(0) \neq E_2(2\pi)$.

Definition 1.38. Let M and N be C^r-manifolds ($r \geq 1$), $f \in C^1(M, N)$ and $p \in M$.

(a) The map f is called *immersive (submersive)* at p if $T_pf : T_pM \to T_{f(p)}N$ is injective (surjective).

(b) The map f is called an *immersion (submersion)* if f is immersive (submersive at every point of M).

(c) Suppose f is a C^r-map, it is called a C^r-embedding if: f is a C^r-submanifold of N and $f : M \to f(M)$ is a C^r-diffeomorphism.

Proposition 1.39. *Let $f \in C^r(M^m, N^n)$ ($r \geq 1$) be injective. Then f is a C^r-embedding iff f is an immersion and $f : M \to f(M)$ is a homeomorphism.*

Proof. $\Rightarrow$ is trivial. For the other direction, first show that $f(M)$ is a C^r-submanifold of N. For any $p \in M$, take local charts (φ, V) and ψ containing p and $f(p)$ on M and N with $\psi(f(p)) = \underline{0}$ and $\varphi(p) = \underline{0}$ respectively. Let $g = \psi \circ f \circ \varphi^{-1} : V \to \mathbb{R}^n$. Then

$$D(f)(\underline{0}) = \left(\frac{\partial g_i}{\partial x_j}(\underline{0})\right), \quad 1 \leq i \leq n, 1 \leq j \leq m$$

has full rank m. By exchanging coordinates, the square matrix

$$\left(\frac{\partial g_i}{\partial x_j}(\underline{0})\right), \quad 1 \leq i, j \leq m$$

is of full rank and therefore invertible. Let $g_{\underline{x}} = (g_1, \cdots, g_m)$ and $g_{\underline{y}} = (g_{m+1}, \cdots, g_n)$, by the inverse function theorem there exists $W \subseteq V$ and $Z \subseteq \mathbb{R}^m$ both open with $\underline{0} \in W \cap Z$ such that $g_x|_W : W \to Z$ is a C^r-diffeomorphism. Let $U = (Z \times \mathbb{R}^{n-m}) \backslash \overline{g(V \backslash W)}$. Then $U \cap g(V \backslash W) = \varnothing$ and claim that $g(W) \subseteq U$.

Assume there exists $w \in W$ such that $g(w) \in \overline{g(V \backslash W)}$, since g is a homeomorphism, there exists a sequence of points $v_n \in V \backslash W$ such that $g(v_n) \to g(w)$ and hence $v_n \to w$. Since $V \backslash W$ is a closed set of V, $w \in V \backslash W$, leading to a contradiction, thus verifying the claim.

Now define $\varphi : U \to \mathbb{R}^n$, $(\underline{z}, \underline{y}) \mapsto (\underline{z}, \underline{y} - g_{\underline{y}}(g_{\underline{x}}^{-1}(\underline{z})))$, then

$$D(\varphi)(\underline{z}, \underline{y}) = \begin{bmatrix} I_m & 0 \\ * & I_{n-m} \end{bmatrix}$$

has full rank. Therefore $\varphi : U \to \varphi(U)$ is a C^r-diffeomorphism, with

$$\varphi^{-1}(\mathbb{R}^m \times \{\underline{0}\}) = \{(\underline{z}, \underline{y}) \in U \mid \exists \underline{w} \in W, g(\underline{w}) = (\underline{z}, \underline{y})\} = g(W) \cap U = g(V) \cap U.$$

therefore $f(M)$ is indeed a submanifold of N. To show that M is diffeomorphic to $f(M)$, notice that f is already a homeomorphism, so one only need to indicate it has local inverse everywhere, whose existence is implied by the inverse function theorem on f. $\square$

Remark 1.40. If M is compact, then the an injective immersion f is an embeddings since $f(M)$ is Hausdorff, and $f : M \to f(M)$ is bijective and continuous, meaning $f : M \to f(M)$ is a homeomorphism already.

Example 1.41. The map $f : S^1 \to S^1 \times S^1, \exp(i\theta) \mapsto (\exp(i\theta), \exp(i2\theta))$ is an embedding, because its derivative is of full rank everywhere and S^1 is compact.

Remark 1.42 (Construction of bump functions). First, construct a function $g : \mathbb{R} \to [0, 1]$,

$$g(t) = \begin{cases} 1, & t \in [-1, 1] \\ 0, & t \in (-\infty, -2] \cup [2, \infty) \\ f(2 - |t|), & t \in [-2, -1] \cup [1, 2] \end{cases}$$

with

$$f(t) = \exp\left(1 - \frac{1}{1 - t^2 \exp\left(1 - \dfrac{1}{t^2}\right)}\right)$$

g is smooth, with $g(t) = 1$ when $|t| \leq 1$ and $g(t) = 0$ when $g(t) \geq 2$. This gives the bump functions $b_n : \mathbb{R}^n \to [0, 1], b_n(\underline{x}) = g(|\underline{x}|_2)$. Also denote $B_s(\underline{x}_0) = \{\underline{x} \in \mathbb{R}^n \mid |\underline{x} - \underline{x}_0|_2 < s\}$.

Theorem 1.43. *Let M be a compact C^r-manifold for some $1 \leq r \leq \infty$, then there exists $g \in \mathbb{N}$ and $f : M \to \mathbb{R}^n$ such that f is a C^r-embedding.*

Proof. Since M is compact, there exists $m \in \mathbb{N}$ and C^r-charts $(\varphi_1, U_1), \cdots, (\varphi_m, U_m)$ such that $B_3(\underline{0}) \subseteq \varphi_i(U_i)$ and $\bigcup_{i=1}^m \varphi_i^{-1}(B_1(\underline{0})) = M$. Define

$$\psi_i(p) = \begin{cases} \varphi_i(p) b_n(\varphi_i(p)), & p \in U_i \\ 0, & p \notin U_i \end{cases}$$

and $f_i : M \to \mathbb{R}^{n+1}$ via $f_i = (\psi_i, b_n \circ \varphi_i)$, and $f : M \to \mathbb{R}^{m(n+1)}, f = (f_1, \cdots, f_m)$. Since f_i is a immersive for all $p \in U_i$, f is an injective immersion, therefore it is an embedding because M is compact. $\square$

Theorem 1.44 (Easy Whitney embedding theorem)**.** *Let M^n be a compact C^r-manifold with $2 \leq r \leq \infty$, then there is a C^r-embedding of M into $\mathbb{R}^{2n+1}$.*

Proof. By theorem 1.43, M embeds in some $\mathbb{R}^q$. If $q \leq 2n+1$, there is nothing to prove. Assume $q > 2n+1$. Replace M by its image under an embedding, therefore M is a C^r-submanifold of $\mathbb{R}^q$. It is sufficient to prove that such an M embeds in $\mathbb{R}^{q-1}$, for repetition of the argument will eventually embed M into $\mathbb{R}^{2n+1}$.

Suppose $M \subset \mathbb{R}^q$, $q > 2n + 1$. Identify $\mathbb{R}^{q-1}$ with $\{x \in \mathbb{R}^q : x_q = 0\}$. For $v \in S^{q-1} \backslash \mathbb{R}^{q-1} \times \{0\}$, let $\pi_v : \mathbb{R}^q \to \mathbb{R}^{q-1}$ be the projection onto $\mathbb{R}^{q-1}$ parallel to v. For the projection to be an injective immersion, choose v such that:

(1) For all $P, Q \in M$, $v \neq \dfrac{P - Q}{|P - Q|}$. This makes sure π_v is injective.

(2) For all $[p, w] \in TM \subset M \times \mathbb{R}^q$, $v \neq \dfrac{w}{|w|}$. This makes sure π_v is an immersion.

Since M is compact, the projection constructed is an embedding. Now need to show that such v does exist. Denote $(TM)_1 = \{[x, \varphi, U, w] \in TM \mid |w|_2 = 1\}$, consider the map σ and ρ:

$$\sigma : M \times M \backslash \Delta \to S^{q-1}, \quad \sigma(P, Q) = \frac{P - Q}{|P - Q|_2}$$

$$\rho : (TM)_1 \to S^{q-1}, \quad [x, \varphi, U, w] \mapsto w$$

since $M \times M \backslash \Delta$ is a manifold of dimension $2n$, and $(TM)_1$ is a manifold of dimension $2n-1$, for $q > 2n+1$, by lemma 1.45, the image of both σ and ρ have empty interior in S^{q-1}, therefore $(\mathbb{R}^{q-1} \times \{0\} \cap S^{q-1}) \cup \operatorname{im}(\sigma) \cup \operatorname{im}(\rho)$ has empty interior, and therefore not equal to S^{q-1}, such v exists. □

Lemma 1.45. *Let $f \in C^1(M, N)$ and $\dim M < \dim N$, then $\operatorname{int}_N(f(M)) = \varnothing$.*

Proof. Recall that in measure theory, any subset $X \subset \mathbb{R}^n$ has measure zero ($\lambda_n(X) = 0$) iff for any $\varepsilon > 0$, there exists $\{C_i\}_{i \in \mathbb{N}}$ a sequence of cubes with $X \subseteq \bigcup_{i=1}^{\infty} C_i$, and $\sum_{i=1}^{\infty} \operatorname{Vol}(C_i) \le \varepsilon$. Extend that definition to subsets of manifolds, i.e. $X \subset M$ (M is a C^1-manifold) has measure zero iff for any charts $(\varphi, U) \in \alpha_1$, $\varphi(U \cap X)$ has measure zero in $\mathbb{R}^n$.

Suppose $f \in C^1(M, N)$ and $\dim M < \dim N$, take charts $\{(\varphi_i, U_i)\}$ on M such that f maps it to a chart in N. For each i, denote $f_i : U \to \mathbb{R}^n$, $f_i = \psi_i \circ f \circ \varphi_i^{-1}$ and

$$g : U \times \mathbb{R}^{n-m} \to \mathbb{R}^n, \quad g(x, y) = f_i(x)$$

then $f_i(U) = g(U \times \{0\})$. Since $U \times \{0\}$ is of measure zero, $f_i(U)$ is of measure zero by lemma 1.48. Therefore $f(M)$ is of measure zero by lemma 1.47. Hence $f(M)$ has empty interior by lemma 1.46. □

Lemma 1.46. *A measure zero set has empty interior.*

Proof. This is trivial since open sets are not of measure zero. □

Lemma 1.47. *$X \subseteq \mathbb{R}^n$ has measure zero iff $\forall x \in X$, $\exists U \subseteq \mathbb{R}^n$ open with $x \in U : \lambda(U \cap X) = 0$.*

Proof. One direction is trivial. For the other direction, for $x \in X$, let U_x the open set such that $x \in U_x$ and $\lambda(U_x \cap X) = 0$. Furthur more, for each $x \in X$, one can find a ball $B_{r_x}(q_x)$ with $r_x \in \mathbb{Q} \cap (0, \infty)$ and $q_x \in \mathbb{Q}^n$ such that $x \in B_{r_x}(q_x) \subseteq U_x$. Suppose $\mathcal{B} = \{B_{r_x}(q_x) \mid x \in X\}$, $\mathcal{B}$ is countable. Since $X = \bigcup_{B \in \mathcal{B}} B \cap X$ and $\lambda_n(B \cap X) = 0$ for each $B \in \mathcal{B}$, X is of measure zero. □

Lemma 1.48. *Let $U \subseteq \mathbb{R}^n$ and $g \in C^1(U, \mathbb{R}^n)$ and $X \subseteq U$ with measure zero. Then $\lambda_n(g(X)) = 0$.*

Proof. By lemma 1.47, one can restrict to the case where $||D(g)(p)||_2 \le K$ and U is a ball. For all $x, y \in U$, one has

$$|g(x) - g(y)|_2 \le K|x - y|_2$$

therefore the image of a cube of edge length l under g is contained in a cube of edge length $lK\sqrt{n}$. If X is covered by cubes C_i with $\sum_{i=1}^{\infty} \operatorname{Vol}(C_i) \le \varepsilon$, then $g(X)$ is covered by cubes C_i' with

$$\sum_{i=1}^{\infty} \operatorname{Vol}(C_i') \le \varepsilon K^n$$

thus $\lambda_n(g(X)) = 0$. □

Definition 1.49. For $\lambda \in L(\mathbb{R}^n, \mathbb{R})$, $H = \{x \in \mathbb{R}^n \mid \lambda(x) \ge 0\}$ is called a *half space*.

Definition 1.50. Let M be a topological space that satisfies conditions listed in remark 1.14. M is called a *manifold with boundary* if $\forall x \in M$, there exists a pair (φ, U) with U open in M and $\varphi : U \to H$ such that $\varphi : U \to \varphi(U)$ is a homeomorphism. $p \in M$ is called a *boundary point* if there exists a chart such that p is mapped to ∂H. The collection of all such points is called the *boundary* of M, denoted by ∂M.

Remark 1.51. The definition of charts, atlases etc. can be generalized to manifolds with boundaries.

Example 1.52. (a) A half space H is a manifold with boundary.

(b) Let $f \in C(\mathbb{R}^{m-1}, \mathbb{R})$ and $M = \{(x, y) \in \mathbb{R}^m\}$, $\varphi : M \to \mathbb{R}^m$, $\varphi(x, y) = (x, y - f(x))$ is a chart for M.

(c) D^2 is a smooth manifold with boundary. Provide an atlas: $\varphi_1 : B_1(0) \to B_1(0), (x, y) \mapsto (x, y)$, $\varphi_2 : \{(x, y) \in D^2 \mid x > 0\} \to \mathbb{R}^2, (x, y) \mapsto (x - \sqrt{1 - y^2}, y)$, $\varphi_3 : \{(x, y) \in D^2 \mid y > 0\} \to \mathbb{R}^2, (x, y) \mapsto (x, y - \sqrt{1 - x^2})$, φ_4 and φ_5 may be constructed similarly.

(d) Let M be a C^∞-manifold with $\partial M = \varnothing$ and $(\varphi, U) \in \alpha_\infty$. Let B be an open ball in $\varphi(U)$, then $M \backslash \varphi^{-1}(B)$ is a manifold with boundary $\partial M = \varphi^{-1}(\partial \overline{B})$.

Definition 1.53. Let U be an open set in a half space H of $\mathbb{R}^n$, and $0 \le r \le \infty$, define $C^r(U, \mathbb{R}^m)$ to be the collection of continuous maps $f : U \to \mathbb{R}^m$ such that all the partial derivatives up to order r are continuous.

Proposition 1.54. *Let H be a half space of $\mathbb{R}^n$, $U \subseteq H$ be open in H, $0 \le r \le \infty$ and $f \in C^r(U, \mathbb{R}^m)$. Then there exists $V \subseteq \mathbb{R}^n$ open with $U \subseteq V$ and $g \in C^r(V, \mathbb{R}^m)$ such that $g|_U = f$.*

Proof. The case $r = \infty$ will not be proved here[4]. For the case where $0 \le r \le \infty$, if $H = \mathbb{R}^n$, then the proof is done. Otherwise w.l.o.g assume $H = \mathbb{R}^{n-1} \times [0, \infty)$. Define

$$V = U \cup \{(\underline{x}, y) \in \mathbb{R}^n \mid (\underline{x}, -\frac{y}{j}) \in U, j = 1, \cdots, r + 1\}$$

then V is open. Define

$$g(\underline{x}, y) = \begin{cases} f(\underline{x}, y), & (\underline{x}, y) \in U \\ \sum_{j=1}^{r+1} c_j f(\underline{x}, -\frac{y}{j}), & (\underline{x}, y) \in V \backslash U \end{cases}$$

such that $c_1, \cdots, c_{r+1} \in \mathbb{R}$ satisfies

$$\sum_{j=1}^{r+1} c_j \left(\frac{-1}{j}\right)^k = 1, \quad k = 0, \cdots, r$$

the determinant of this linear equation (Van der Monde determinant) is non-zero, therefore such c_i exists. ☐

Example 1.55. For $f \in C^1((-1, 0], \mathbb{R})$, $f(x) = (1 + x)^2$, proposition 1.54 says

$$g(x) = \begin{cases} (1 + x)^2, & x \in (-1, 0] \\ (1 + x)^2 - 3x^2, & x \in (0, 1) \end{cases}$$

is a C^1 extension.

Definition 1.56. $N \subseteq \mathbb{R}^n$ is called a C^r*-submanifold* of dimension k if for all $p \in N$, there exists $(\varphi, U) \in \alpha_r$ with $p \in U$ and a half space H of $\mathbb{R}^k$ such that $\varphi^{-1}(H \times \{0\}) = N \cap U$. For $N \subseteq M$ where M is a C^r-manifold, N is said to be a k-dimensional C^r*-submanifold* if for all $p \in N$, there exists $(\varphi, U) \in \alpha_r$ with $p \in U$ such that $\varphi(U \cap N)$ is a C^r-submanifold of $\mathbb{R}^m$.

Definition 1.57. Let N^n be a C^r-submanifold of M^m. Then N is said to be *neat* if

(1) $\partial N = (\partial M) \cap N$;

(2) For all $p \in N$, there exists $(\varphi, U) \in \alpha_r$ such that $N \cap U = \varphi^{-1}(\mathbb{R}^n \times \{0\})$.

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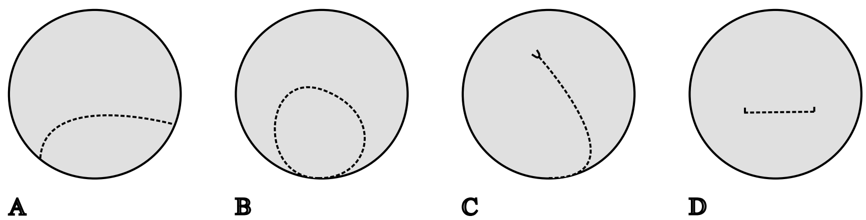
[4]It is done in "Analytic extensions of differential functions defined in closed sets" by Whitney.

Example 1.58. Consider the following C^1-submanifold of $\overline{B_1}(0)$ shown as (A), (B), (C), (D). The submanifolds are indicated with dash lines. The submanifolds contain the point where it intersects the boundary of $\overline{B_1}(0)$. In (C), the other end of the line is open; in (D), the line is closed. Only (A) shows a neat submanifold. (B) fails both two requirements in definition 1.57; (C) only fails definition 1.57 (2); (D) only fails definition 1.57 (1).



Definition 1.59. A C^r-embedding is called *neat* if the image is a neat C^r-submanifold.

Example 1.60. The embedding from $[-1, 1]$ to the submanifold in graph (A) is a neat embedding.

Theorem 1.61. *Let M^n be a C^r-compact manifold, then there exists a neat C^r-embedding into $\mathbb{R}^{2n} \times [0, \infty)$.*

2. Approximation

Definition 2.1. Let X be a topological space and $\mathcal{A} = (A_i)_{i\in I}$ be a family of subsets of X.

(a) $\mathcal{A}$ is called a *covering* of X if $X = \bigcup_{i\in I} A_i$. If all A_i is open, then it is called an *open covering.*

(b) $\mathcal{A}$ is called *locally finite* if for all $x \in X$, there exists U an open neighbourhood of X such that

$$|\{i \in I \mid A_i \cap U \neq \varnothing\}| < \infty$$

(c) Let $\mathcal{A}$ and $\mathcal{B} = (B_j)_{j\in J}$ be coverings of X. $\mathcal{B}$ is called a *refinement* of $\mathcal{A}$ if for all $j \in J$, $\exists i \in I$ such that $B_j \subseteq A_i$, and $\mathcal{B}$ is called a *shrinking* of $\mathcal{A}$ if $J = I$ and $\overline{B_j} \subseteq A_j$.

Definition 2.2. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of a topological space X. A family $\Lambda = (\lambda_i)_{i\in I}$ where $\lambda_i \in C(X, [0, 1])$ is called a *partition of unity* subordinated to $\mathcal{U}$ if

(a) $\forall i \in I$, $\operatorname{supp}(\lambda_i) = \overline{\{x \in X \mid \lambda_i(x) \neq 0\}} \subset U_i$;

(b) $\operatorname{supp}(\lambda_i)_I$ is locally finite;

(c) $\forall x \in X'$, $\sum_{i\in I} \lambda_i(x) = 1$.

Remark 2.3. Partition of unity provides a way to build a global function out of local components. Also, given an open cover $\mathcal{A}$ with a partition of unity Λ, then $(\operatorname{int}(\operatorname{supp}(\lambda_i)))_I$ is a locally finite cover.

Proof. $\forall x \in X$, since $\sum_{i\in I} \lambda_i(x) = 1$, there exists $i_0 \in I$ such that $\lambda_{i_0}(x) > 0$. Therefore $x \in \lambda_{i_0}^{-1}((0, 1]) \subseteq \operatorname{int}(\operatorname{supp}(\lambda_i))$. Since $(\operatorname{supp}(\lambda_i))_I$ is locally finite, $\operatorname{int}(\operatorname{supp}(\lambda_i))_I$ is locally finite. □

Example 2.4. (a) Suppose $X = \mathbb{R}$, $\mathcal{A} = \{\mathbb{R}\}$, then $\Lambda = \{1_{\mathbb{R}}\}$ is a partition of unity.

(b) Suppose $X = S^1$, $U_1 = S^1\backslash\{-1\}$, $U_2 = S^1\backslash\{1\}$, $\mathcal{A} = \{U_1, U_2\}$. Then let

$$\widetilde{\lambda_1}(\exp(i\theta)) = \exp\left(\frac{1}{(\theta - \pi/4)(\theta - 7\pi/4)}\right), \quad \theta \in (\pi/4, 7\pi/4) \quad \text{(takes 0 otherwise)}$$

and

$$\widetilde{\lambda_2}(\exp(i\theta)) = \exp\left(\frac{1}{(\theta - 3\pi/4)(\theta + 3\pi/4)}\right), \quad \theta \in (-3\pi/4, 3\pi/4) \quad \text{(takes 0 otherwise)}$$

Suppose for $x \in S^1$,

$$\lambda_i(x) = \frac{\lambda_i(x)}{\widetilde{\lambda_1}(x) + \widetilde{\lambda_2}(x)}$$

then the support of λ_i is just $\operatorname{supp}(\widetilde{\lambda_i}(x)) \subseteq U_i$, and since $\lambda_1 + \lambda_2 = 1$, they constitutes a partition of unity.

Theorem 2.5. *Let M be a C^r-manifold, then every open cover of M has subordinate partition of unity.*

Proof. Let $\mathcal{U} = (U_i)_I$ be an open cover of M. By lemma 2.7, one may take $(\varphi_\alpha, V_\alpha)_{\alpha\in A}$ a locally finite atlas such that $(\overline{V_\alpha})_{\alpha\in A}$ refines $\mathcal{U}$ and $\varphi_\alpha(V_\alpha) \subset \mathbb{R}^n$ is bounded and each $\overline{V_\alpha}$ is compact. By lemma 2.8, there is a shrinking $\{W_\alpha\}_{\alpha\in A}$ of $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$, and each $\overline{W_\alpha} \subset V_\alpha$ is compact. By lemma 2.6, it suffices to find a C^r partition of unity subordinate to $\mathcal{V}$.

For each $\alpha \in A$, cover the compact set $\varphi_\alpha(\overline{W_\alpha})$ by a finite number of closed balls $B(\alpha, 1), \cdots, B(\alpha, k(\alpha))$ contained in $\varphi_\alpha(V_\alpha)$. Choose C^∞ bump functions $\lambda_{\alpha,j} : \mathbb{R}^n \to [0, 1]$ for $j = 1, \cdots, k(\alpha)$ such that $\lambda_{\alpha,j}(x) > 0$ iff $x \in \operatorname{int} B(\alpha, j)$. Put

$$\lambda_\alpha = \sum_{j=1}^{k(\alpha)} \lambda_{\alpha,j} : \mathbb{R}^n \to [0, \infty)$$

then $\lambda_\alpha(x) > 0$ if $x \in \varphi_\alpha(\overline{W_\alpha})$ and $\lambda_\alpha(x) = 0$ if $x \in \mathbb{R}^n\backslash \bigcup_j B(\alpha, j)$. Put $\mu_\alpha : M \to [0, \infty)$:

$$\mu_\alpha(x) = \begin{cases} \lambda_\alpha(\varphi_\alpha(x)), & x \in V_\alpha \\ 0, & x \in M\backslash V_\alpha \end{cases}$$

then μ_α is C^r, $\mu_\alpha > 0$ on $\overline{W_\alpha}$, and $\operatorname{supp}\mu_\alpha \subset V_\alpha$. Then $\nu_\alpha = \mu_\alpha / \sum_\alpha \mu_\alpha$ is a partition of unity on $\mathcal{V}$. $\square$

Lemma 2.6. *Let $\mathcal{B}$ and $\mathcal{A}$ be open covers of X such that $\mathcal{A}$ refines $\mathcal{B}$. Then $\mathcal{B}$ has a partition of unity if $\mathcal{A}$ has one.*

Proof. Let $(\lambda_i)_I$ be a partition of unity subordinated to $\mathcal{A}$. Since $\mathcal{A}$ refines $\mathcal{B}$, suppose a map $f : I \to J$ such that $A_i \subseteq B_{f(i)}$. Put $M_j = \sum_{i \in f^{-1}(j)} \lambda_j$. Claim that M_j is a partition of unity on $\mathcal{B}$.

Since $\operatorname{supp}(M_j) \subseteq \bigcup_{i\in f^{-1}(j)} \operatorname{supp}(\lambda_i) \subseteq \bigcup_{i\in f^{-1}(j)} A_i \subseteq B_j$ and

$$\sum_{j\in J} M_j(x) = \sum_{j\in J} \sum_{i\in f^{-1}(j)} \lambda_i(x) = \sum_{i\in I} \lambda_i(x) = 1$$

M_j satisfies definition 2.2 (a) (c). For definition 2.2 (b), note that there exists an open neighbourhood U for any $x \in X$ such that $S = \{i \in I \mid U \cap \operatorname{supp}(\lambda_i) \neq \varnothing\}$ is finite. Take $j \in J$ such that $\operatorname{supp}(M_j) \cap U \neq \varnothing$, then $\exists i \in I$ such that $f(i) = j$ and $\operatorname{supp}(\lambda_i) \cap U \neq \varnothing$. Thus $i \in S$, and therefore $j = f(i) \in f(S)$, where $f(S)$ is finite. $\square$

Lemma 2.7. *Suppose $\mathcal{U} = (U_i)_I$ is an open cover of M. Then there is a locally finite atlas $(\varphi_\alpha, V_\alpha)_{\alpha\in A}$ such that:*

(1) $(\overline{V_\alpha})_{\alpha\in A}$ refines $\mathcal{U}$;

(2) $\varphi_\alpha(V_\alpha) \subset \mathbb{R}^n$ is bounded and $\overline{V_\alpha} \subset M$ is compact for each $\alpha \in A$.

Proof. For each $x \in M$, suppose $x \in (U_i, \varphi_i)$, then there exists ε such that $B_\varepsilon(\varphi_i(x)) \subset \varphi_i(U_i)$. Then put $W_{x,i} = \varphi_i^{-1}(B_{\varepsilon/2}(\varphi_i(x)))$ and $\mathcal{W} = (W_{x,i})_{M\times I}$, it is clear that $\mathcal{W}$ covers M. Since M is paracompact, there is a locally finite refinement $\mathcal{V}$ of $\mathcal{W}$. It is easy to verify that $\mathcal{V}$ is the cover required. $\square$

Lemma 2.8 (Shrinking lemma)**.** *Let X be a T_4 topological space, and let $(U_i)_I$ be a point finite open cover. Then it has a shrinking.*

Proof. Consider the set S of pairs $(J, \mathcal{V})$ consisting of a subset $J \subset I$ and an I-indexed set of open subsets $\mathcal{V} = \{V_i\}_I$ with the property that:

(1) $(i \in J \subset I) \Rightarrow (\overline{V_i} \subset U_i)$;

(2) $(i \in I \backslash J) \Rightarrow (V_i = U_i)$;

(3) $\{V_i\}_{i\in I}$ is an open cover of X.

Equip the set S with a partial order $\le$ by setting

$$(J_1, \mathcal{V}) \le (J_2, \mathcal{W}) \Leftrightarrow (J_1 \subset J_2, \forall_{i\in J_1}(V_i = W_i))$$

then an element of $(S, \le)$ with $J = I$ would be the shrinking required. First, claim that a maximal element of $(S, \le)$ has $J = I$. For assume on contrary that there were $i \in I\backslash J$. By lemma 2.9, one may replace that single V_i with a smaller open set V_i' to obtain $\mathcal{V}'$, then $(J, \mathcal{V}) < (J \cup \{i\}, \mathcal{V}')$, contradiction.

Now show that the maximal element exists. By Zorn's lemma, one need to check that every totally ordered subset in $(S, \le)$ has an upper bound. Let $T \subset S$ be one such subset. Suppose $K = \bigcup_{(J,\mathcal{V})\in T} J$, and define $\mathcal{W} = (W_i)_I$ as following:

(1) For $i \in K$, pick any $(J, \mathcal{V})$ in T with $i \in J$ and set $W_i = V_i$. This is well defined by the assumption that T is totally ordered.

(2) For $i \in I\backslash K$ define $W_i = U_i$.

If $(K, \mathcal{W}) \in S$, then it is a upper bound of T by construction. Thus it remains to show that $(K, \mathcal{W}) \in S$, i.e. $(W_i)_I$ is a cover of X.

Take any $x \in X$, for all $t = (J, \mathcal{V}) \in T$, denote $J = J_t$, $\mathcal{V} = \mathcal{V}_t = (V_t^i)_I$. Suppose $S_x(t) = \{i \in I \mid x \in V_t^i\}$. Since $(U_i)_I$ is a point finite cover, $\mathcal{V}_t$ is also a point finite cover, therefore $0 < |S_x(t)| < \infty$. It is also clear that if $t_1 \le t_2$, then $S_x(t_2) \subseteq S_x(t_1)$. By lemma 2.10, $\bigcap_{t\in T} S_x(t) \neq \varnothing$. Since W_i can be written as

$W_i = \bigcap_{t\in T} V_t^i$, it immediately follows that $\bigcap_{t\in T} S_x(t) = \{i \in I \mid x \in W_i\} \neq \varnothing$. Therefore $x \in \cup_{i\in I} W_i$, hence $(K, \mathcal{W})$ is indeed an element of S. $\square$

Lemma 2.9. *Let X be a T_4 topological space and let $\{U_1, U_2\}$ be an open cover. Then there exists an open set $V_1 \subset X$ whose closure is contained in U_1 and such that $\{V_1, U_2\}$ is still an open covering of X.*

Proof. Since $X = U_1 \cup U_2$, $X\backslash U_i$ are disjoint closed subsets. Since X is T_4, there exist disjoint open sets $X\backslash U_2 \subset V_1$ and $X\backslash U_1 \subset V_2$, and $V_1 \subset X\backslash V_2 \subset U_1$. Since $X\backslash V_2$ is closed, $\overline{V_1} \subset U_1$. $\square$

Lemma 2.10. *Suppose a totally ordered set T, and sets S_t for $t \in T$ such that: (1) $0 < |S_t| < \infty$ for each $t \in T$; (2) $S_{t_2} \subseteq S_{t_1}$ if $t_1 \leq t_2$; then $\bigcap_{t\in T} S_t \neq \varnothing$.*

Proof. Claim that there exists t_0 such that $|\bigcap_{t\in T} S_t| = |S_{t_0}|$. For assume on contrary that such t_0 does not exist. Then for any $t_1 \in T$, there always exists $t_2 \in T$ with $t_2 \geq t_1$ such that $|S_{t_2}| < |S_{t_1}|$. Since $0 < |S_{t_1}| < \infty$, contradiction. $\square$

Definition 2.11. Suppose $0 \leq r < \infty$. The *weak topology* on $C^r(M, N)$ is the coarest topology containing the sets $N^r(f, (\varphi, U), (\psi, V), K, \varepsilon)$ for $f \in C^r(M, N), (\varphi, U) \in \alpha_r, (\psi, V) \in \beta_r$ such that $K \subseteq U, f(K) \subset V$ and $\varepsilon > 0$. $N^r(f, (\varphi, U), (\psi, V), K, \varepsilon)$ is a collection of $g \in C^r(M, N)$ such that

(1) $g(K) \subset V$;

(2) $||D^{(k)}(\psi \circ f \circ \varphi^{-1})(\underline{x}) - D^{(k)}(\psi \circ g \circ \varphi^{-1})(\underline{x})||_k < \varepsilon$ for all $\underline{x} \in \varphi(K)$ and $k = 0, \cdots, r$.

Remark 2.12. (a) Recall the norm $||\cdot||_k : L(\mathbb{R}^m, L(\mathbb{R}^m, \cdots, L(\mathbb{R}^m, \mathbb{R}^n))) \to \mathbb{R}$ can be taken as

$$||F||_k = \sum_{(i_1,\cdots,i_k)\in\{1,\cdots,m\}_k} |F(e_{i_1})(e_{i_2})\cdots(e_{i_k})|$$

for example, for $f \in C^2(\mathbb{R}^m, \mathbb{R})$,

$$||D^{(0)}(f)(\underline{x})||_0 = |f(\underline{x})|, \quad ||D^{(1)}(f)(\underline{x})||_1 = \sum_{i=1}^m \left|\frac{\partial f}{\partial x_i}(\underline{x})\right|, \quad ||D^{(2)}(f)(\underline{x})||_2 = \sum_{i=1}^m \sum_{j=1}^n \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x})\right|$$

(b) Suppose $M = \mathbb{R}$, $f \in C^2(\mathbb{R}, \mathbb{R})$, $f(x) = x^2$, $g(x) = x^2 + x^3/1000$, $K = [-1, 1]$, and $(\varphi, U) = (\psi, V) = (\mathrm{id}, \mathbb{R})$. Then $g \in N^2(f, K, 1/100)$ but $g \notin N^2(f, K, 1/200)$.

(c) The weak topology does not control the behavior at infinity, since it only concerns compact sets.

Definition 2.13. The *strong topology* is the coarest topology containing the sets

$$N^r(f, \Phi, \Psi, K, \varepsilon) = \bigcap_{i\in I} N^r(f, (\varphi_i, U_i), (\psi_i, V_i), K_i, \varepsilon_i)$$

where $\Phi = (\varphi_i, U_i)_I$ is a locally finite family of C^r-charts of M, $\Psi = (\psi_i, V_i)_I$ is a set of C^r-charts of N, $K = (K_i)_I$ with $K_i \subseteq U_i$ and compact, $\varepsilon = (\varepsilon_i)_I$ with $\varepsilon_i > 0$, and $f \in C^r(M, N)$ with $f(K_i) \subseteq V_i$ for all $i \in I$.

Remark 2.14. Denote the weak and strong topology on $C^r(M, N)$ with $C^r_w(M, N)$ and $C^r_s(M, N)$. The $r = \infty$ and $r = w$ case will be defined in definition 2.16.

Example 2.15. (a) Suppose $M = N = \mathbb{R}$, $\delta > 0$, $I = \mathbb{Z}$, $(\varphi_i, U_i) = (\mathrm{id}, (2i, 2i+1))$, $(\psi_i, V_i) = (\mathrm{id}, \mathbb{R})$, $K_i = [2i + \delta, 2i + 1 - \delta]$, $f \in C^r(M, N)$, and $\varepsilon = (\varepsilon_i)_{\mathbb{Z}}$ where $\varepsilon_i \to 0$. Let g be the function shown in the following graph, then $g \in N^0(f, \Phi, \Psi, K, \varepsilon)$ but $g \notin N^1(f, \Phi, \Psi, K, \varepsilon)$, since $\sup_{x\in K_{-1}} |g'(x) - f'(x)| \geq \varepsilon_{-1}$.

(b) $C^0_s(\mathbb{R}, \mathbb{R})$ and $C^0_w(\mathbb{R}, \mathbb{R})$ are not the same as $C^0_{||\cdot||_\infty}(\mathbb{R}, \mathbb{R})$. To see this, consider $f_n(x) = 1/n$, then $f_n \to 0$ in $C^0_{||\cdot||_\infty}(\mathbb{R}, \mathbb{R})$, but $f_n \not\to 0$ in $C^0_s(\mathbb{R}, \mathbb{R})$ since for all $n \in \mathbb{N}$, $f_n \notin N(0, \Phi, \Psi, K, \varepsilon)$ where $(\varphi_i, U_i) =$

$(\mathrm{id}, (2i, 2i+1))$, $(\psi_i, V_i) = (\mathrm{id}, \mathbb{R})$ and $K_i = [2i + 1/4, 2i + 3/4]$, $\varepsilon_i = 1/(1 + |i|)$. For the weak topology, construct a sequence of bump functions that supported by $[n, n + 1]$.

(c) If $\partial M = \partial N = \varnothing$, then $\mathrm{emb}^r(M, N)$ is open in $C^r_s(M, N)$, but wrong for the weak topology.

Definition 2.16. The *weak topology* on $C^\infty(M, N)$ is defined to be the coarest topology such that all the inclusions $C^\infty(M, N) \hookrightarrow C^r(M, N)$ for $0 \le r < \infty$ are continuous. The *strong topology* on $C^\infty(M, N)$ is defined similarly. The weak and strong topology on $C^w(M, N)$ is the subspace topology induced from $C^\infty(M, N)$.

Proposition 2.17. *Let M, N be smooth manifolds and M be compact. Then the strong and weak topology on $C^r(M, N)$ coincide.*

Proof. If $U \subset C^r(M, N)$ is open in the weak topology, it is open in the strong topology by definition. For the inverse implication, suppose a base for the strong topology $N^r(f, \Phi, \Psi, K, \varepsilon)$. Since $\Phi = (\varphi_i, U_i)_I$ is a locally finite family of charts, for any $x \in M$, there exists N_x an open neighbourhood of x such that $I_x = \{i \in I \mid N_x \cap U_i \neq \varnothing\}$ is finite. Since M is compact, there exists $x_1, \cdots, x_n$ such that $M = \bigcup_{i=1}^n N_{x_i}$. Then $I = \bigcup_{i=1}^n I_{x_n}$ is also finite. Therefore $N^r(f, \Phi, \Psi, K, \varepsilon) = \bigcap_{i=1}^n N^r(f, (\varphi_i, U_i), (\psi_i, V_i), K_i, \varepsilon_i)$ is an open set in the weak topology. □

Proposition 2.18. *Suppose $0 \le r < t$, M, N smooth manifolds and U is an open subset of $C^r_s(M, N)$, then $C^t(M, N) \cap U$ is open in $C^t_s(M, N)$.*

Proof. For all $f \in C^t_s(M, N) \cap U$, since U is open in $C^r_s(M, N)$, there exists $N^r(f, \Phi, \Psi, K, \varepsilon) \subset U$. By

$$N^t(f, \Phi, \Psi, K, \varepsilon) \subset N^r(f, \Phi, \Psi, K, \varepsilon) \cap C^t_s(M, N)$$

one gets $N^t(f, \Phi, \Psi, K, \varepsilon) \subset U \cap C^t_s(M, N)$. □

Remark 2.19. For $r \ge 1$, denote $\mathrm{imm}^r(M, N)$, $\mathrm{subm}^r(M, N)$, $\mathrm{emb}^r(M, N)$, $\mathrm{emb}^r_c(M, N)$, $\mathrm{diff}^r(M, N)$ to be the set of C^r immersions, submersions, embeddings, closed embeddings and diffeomorphisms from M to N respectively.

Theorem 2.20. *Suppose M, N are C^r-manifolds and $r \ge 1$, then:*
(a) $\mathrm{imm}^r(M, N)$ is open in $C^r_s(M, N)$.
(b) $\mathrm{subm}^r(M, N)$ is open in $C^r_s(M, N)$.
(c) Suppose $\partial M = \partial N = \varnothing$, then $\mathrm{diff}^r(M, N)$ is open in $C^r_s(M, N)$.

Proof. (a) Since $\mathrm{imm}^r(M, N) = \mathrm{imm}^1(M, N) \cap C^r(M, N)$, it suffices to prove that $\mathrm{imm}^1(M, N)$ is open. Suppose $f : M \to N$ is a C^1 immersion, choose a neighbourhood $N^1(f, \Phi, \Psi, K, \varepsilon)$ as follows. Let $\Psi^0 = \{\psi_\beta, V_\beta\}_{\beta \in B}$ be any atlas for N. Pick an atlas $\Phi = \{\varphi_i, U_i\}_{i \in I}$ for M so that each U_i has compact closure, and for each $i \in I$ there exists $\beta_i \in B$ such that $f(U_i) \subseteq V_{\beta_i}$. Put $V_{\beta_i} = V_i$, $\psi_{\beta_i} = \psi_i$ and $\Phi = \{\psi_i, V_i\}_{i \in I}$. Let $K = (K_i)_{i \in I}$ be a compact cover of M with $K_i \subset U_i$. Endow the set $L(\mathbb{R}^m, \mathbb{R}^n)$ with the topology induced by the metric $||\cdot||_1$. Then set

$$A_i = \{D(\psi_i \circ f \circ \varphi_i^{-1})(x) \in L(\mathbb{R}^m, \mathbb{R}^n) \mid x \in \varphi_i(K_i)\}$$

is a compact set since the map $f : K_i \to A_i, x \mapsto D(\psi_i \circ f \circ \varphi_i^{-1})(x)$ is a continuous surjective map. Denote I to the set of all injective linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$, then I is open in $L(\mathbb{R}^m, \mathbb{R}^n) = L$, and $A_i \subset I$.

Claim that $d(A_i, L\backslash I) > 0$. For assume on contrary $d(A_i, L\backslash I) = 0$. Then for all $\varepsilon > 0$, there exists $a \in A_i$ and $t \in L\backslash I$ such that $0 < d(a, t) < \varepsilon$. Take $\varepsilon_n = 1/n$ for $n \in \mathbb{N}^+$, then for each n there are $a_n \in A_i$ and $t_n \in L\backslash I$ such that $0 < d(a_n, t_n) < 1/n$. Since A_i is compact, the sequence a_n has a convergent subsequence a_{n_i} for $i \in \mathbb{N}^+$. Denote $a_0 = \lim_{i \to \infty} a_{n_i}$, then $0 \le d(a, t_{n_i}) \le d(a, a_{n_i}) + d(a_{n_i}, t_{n_i})$, hence $d(a, t_{n_i}) \to 0$ as $i \to \infty$. Thus $\lim_{i \to \infty} t_{n_i} = a$. Since $L\backslash I$ is closed, $a \in L\backslash I$, contradiction.

Since $d(A_i, L\backslash I) > 0$, take ε_i such that $d(A_i, L\backslash I) > \varepsilon_i$. Then for all $T \in L$, if $||T - S|| < \varepsilon_i$ for some $S \in A_i$, then $T \in I$. Let $\varepsilon = (\varepsilon_i)_I$, then $N^r(f, \Phi, \Psi, K, \varepsilon) \subset \mathrm{imm}^1(M, N)$.

The same argument goes for (b). For (c), observe that when $\partial M = \partial N = \varnothing$, $\mathrm{diff}^r = \mathrm{emb}^r_c(M, N) \cap \mathrm{subm}^r(M, N)$ is a intersection of two open maps by remark 2.21 (b). $\square$

Remark 2.21. (a) Theorem 2.20 (c) is wrong if manifolds with boundaries are considered. Take $M = N = [0, 1]$, $f = \mathrm{id}_M$ and $g_\delta(x) = \delta + x(1-2\delta)$. Then $\lim_{\delta\to 0} g_\delta = f \in C^1_s(M, N)$. But g_δ is not a diffeomorphism. From now on, only consider manifolds without boundary.

(b) $\mathrm{emb}^r(M, N)$, $\mathrm{emb}^r_c(M, N)$ are also open in $C^r_s(M, N)$. But I have not had time to look into it.

Definition 2.22. Let $U \subseteq \mathbb{R}^m$ open and $\sigma > 0$ such that U contains a closed ball of radius σ. Suppose a bump function $\theta \in C^\infty(\mathbb{R}^m, [0, \infty))$ such that $\operatorname{supp}\theta \subset \overline{B_\sigma}(\underline{0}) \subset \mathbb{R}^m$ and $U_\sigma = \{\underline{x} \in U \mid \overline{B_\sigma}(\underline{x}) \subset U\}$. Define a map $C^0(U, \mathbb{R}^n) \to C^\infty(U_\sigma, \mathbb{R}^n)$ via $f \mapsto \theta * f$, where

$$(\theta * f)(\underline{x}) = \int_{\overline{B_\sigma}(\underline{x})} \theta(\underline{x} - \underline{y}) f(\underline{y}) d\underline{y}$$

the map $\theta * f$ is called *convolution* of θ with f. A map $\theta \in C^\infty(\mathbb{R}^m, [0, \infty))$ with support in $\overline{B_\sigma}(\underline{0})$ ($\sigma > 0$) is called a *convolution kernel* if $\int_{\mathbb{R}^m} \theta(\underline{y}) d\underline{y} = 1$.

Remark 2.23. (a) Let $f \in C^r(U, \mathbb{R}^n)$ and $K \subset U$ be compact. Denote

$$||f||_{r,k} = \sup\{||D^k(f)(\underline{x})|| \mid \underline{x} \in K, k = 0, \cdots, r\}$$

(b) In analysis it is shown that $D^k(\theta * f) = D^k(\theta) * f$.

(c) Suppose $f \in C^k(U, \mathbb{R}^m)$, then $D^k(\theta * f) = \theta * D^k(f)$ because by substituting $\underline{z} = \underline{x} - \underline{y}$, one obtains

$$(\theta * f)(\underline{x}) = \int_{\overline{B_\sigma}(\underline{0})} \theta(\underline{z}) f(\underline{x} - \underline{z}) d\underline{z}$$

Proposition 2.24. *Let $U \subseteq \mathbb{R}^m$ be open, nonempty, $K \subset U$ compact and $f \in C^r(U, \mathbb{R}^n)$ with $0 \le r \le \infty$. Suppose $\varepsilon > 0$, then there exists $\sigma > 0$ such that: (1) $K \subset U_\sigma$; (2) for all convolution kernel θ with* $\operatorname{supp}(\theta) \subset \overline{B_\sigma}(\underline{0})$, $||\theta * f - f||_{r,k} < \varepsilon$.

Proof. Suppose W open such that $K \subset W \subset \overline{W} \subset U$ where $\overline{W}$ is compact by taking W to be the union of finitely many open balls. Since $f|_{\overline{W}}$ is continuous and $\overline{W}$ is compact, f is uniformly continuous on $\overline{W}$. Then there exists $\sigma > 0$ such that for all $\underline{x}, \underline{y} \in \overline{W}$ with $|\underline{x} - \underline{y}|_2 < \sigma$, one has $|f(\underline{x}) - f(\underline{y})|_2 < \varepsilon/2$. Take σ smaller than $d(K, U\backslash W)$ and $\underline{x} \in K$, then

$$\begin{aligned}
|\theta * f(\underline{x}) - f(\underline{x})|_2 &= \left| \int_{\overline{B_\sigma}(\underline{0})} \theta(\underline{y}) f(\underline{x} - \underline{y}) d\underline{y} - f(x) \int_{\overline{B_\sigma}(\underline{0})} \theta(\underline{y}) d\underline{y} \right|_2 \\
&= \left| \int_{\overline{B_\sigma}(\underline{0})} \theta(\underline{y}) (f(\underline{x} - \underline{y}) - f(\underline{x})) d\underline{y} \right|_2 \\
&\le \int_{\overline{B_\sigma}(\underline{0})} \theta(\underline{y}) \left| (f(\underline{x} - \underline{y}) - f(\underline{x})) \right|_2 d\underline{y} \le \varepsilon/2
\end{aligned}$$

Thus the proposition is true for $k = 0$. For $k \ge 1$, since $D^k(\theta * f) = \theta * D^k(f)$, the same argument applies. $\square$

Theorem 2.25. *Let M and N be C^s-manifolds with $1 \le s \le \infty$, then $C^s(M, N)$ is dense in $C^r_s(M, N)$ for $0 \le r < s$.*

Sketch proof. Unfinished! $\square$

Theorem 2.26. *Let (M, α_r) be a C^r-manifold with $1 \leq r < \infty$. For every s, $r < s \leq \infty$, there exists a compactible C^s-differential structure $\beta \subset \alpha_r$, and β is unique up to C^s-diffeomorphism.*

Proof. First show uniqueness. Let β and γ be C^s-differential structures in α_r. Then $\operatorname{diff}^r((M,\beta),(M,\gamma)) = \operatorname{diff}^r((M,\alpha_r),(M,\alpha_r))$. Since $\mathrm{id} \in \operatorname{diff}^r((M,\alpha_r),(M,\alpha_r))$, $\operatorname{diff}^r((M,\beta),(M,\gamma)) \neq \varnothing$. By theorem 2.20, $\operatorname{diff}^r(M,M)$ is open in $C^r_s(M,M)$. By theorem 2.25, $C^s((M,\beta),(M,\gamma))$ is dense in $C^r_s(M,M)$. Therefore $\operatorname{diff}^s((M,\beta),(M,\gamma)) = \operatorname{diff}^r(M,M) \cap C^s((M,\beta),(M,\gamma)) \neq \varnothing$.

For convenience denote a differential structure and its restriction to an open set the same symbol. By Zorn's lemma there is a nonempty open set $B \subset M$ and a C^s differential structure $\beta \subset \alpha_r$ on B such that (B,β) is maximal in the partial order given by inclusions. To show the existence of a C^s-differential structure, claim that $B = M$.

For assume on contrary $B \neq M$. Then there exists a chart $(\varphi, U) \in \alpha_r$ such that $U \cap (M \backslash B) \neq \varnothing$. If $U \cap B = \varnothing$, then $\beta \cup \{(\varphi, U)\}$ is a C^s-atlas and $B \subsetneq B \cup U$, contradicting the assumption that (B,β) is maximal.

So $W = B \cap U \neq \varnothing$. Then $W \subseteq U$ open and there exists $N \subseteq C^r_s(W, \varphi(W))$ open such that $T : N \to C^r_s(U, \varphi(U))$ is continuous by lemma 2.27 (with $f = \varphi$ here). Thus $N' = T^{-1}(\operatorname{diff}^r(U, \varphi(U)))$ is open. By the definition of T, $N' \subseteq \operatorname{diff}^r(W, \varphi(W))$. Since $\varphi \in N'$, N' is nonempty. Since $C^s(W, \varphi(W))$ is dense in $C^r_s(W, \varphi(W))$, $N' \cap C^s(W, \varphi(W)) \neq \varnothing$. Suppose $\varphi_0 \in N' \cap C^s(W, \varphi(W))$, then $\beta \cup \{(U, T(\varphi_0))\}$ is a C^s-atlas and $B \subsetneq B \cup U$, contradiction. $\square$

Lemma 2.27. *Let U be a C^r-manifold, $0 \leq r < \infty$, and $W \subset U$ an open set. Let $V \subset \mathbb{R}^n$ be open, $f \in C^r_s(U,V)$, and put $f(W) = V'$. Then there is a neighbourhood $N \subset C^r_s(W, V')$ of $f|_W$ such that if $g_0 \in N$, the map*

$$T(g_0) = g : U \to V$$

$$g(x) = \begin{cases} g_0(x), & x \in W \\ f(x), & x \in U \backslash W \end{cases}$$

is C^r, and $T : N \to C^r_s(U,V)$ is continuous.

Proof. Let $(\varphi_i, U_i)_I$ be a locally finite family of charts of U which covers $\operatorname{bd} W$ the boundary of W in U. W.l.o.g assume that every $\overline{U_i}$ is compact. Choose a shrinking $(L_i)_I$ of $(U_i)_I$. Define $N \subset C^r_s(W, V')$ as follows:

$$N = \{h \in C^r(W, V') \mid \forall_{i \in I} \forall_{y \in \varphi_i(L_i \cap W)} \forall_{k=0,\cdots,r} ||D^k(h \circ \varphi_i^{-1})(y) - D^k(f \circ \varphi^{-1})(y)||_k < d(y, \varphi_i(U_i \backslash W))\}$$

claim that N is an open neighbourhood of $f|_W$. Since $(L_i)_I$ is locally finite, take $(K_w)_W$ such that each K_w meets only finitely many L_i, and then replace it with a shrinking. Since $\{K_\beta\} \cup \{U\}$ is a cover of U, by paracompactness W has a locally finite open cover $\{K_\alpha\}$ such that each $\overline{K_\alpha}$ meets only finitely many L_i. Then the map $d : \overline{K_\alpha} \cap L_i \to \mathbb{R}, x \mapsto d(\varphi_i(x), \varphi_i(U_i \backslash W))$ is bounded away from 0, since (1) $\overline{K_\alpha} \cap \overline{L_i}$ is disjoint from $U_i \backslash W$, (2) $U_i \backslash W$ is closed in U_i, and $\overline{K_\alpha} \cap \overline{L_i}$ is compact in U_i, and then metric argument in 2.20 (a) applies. Thus N is indeed an open neighbourhood.

Now show that the g is C^r. It suffices to prove that $\lambda_i : \varphi_i(U_i) \to \mathbb{R}^n$

$$\lambda_i(x) = \begin{cases} h \circ \varphi_i^{-1}(x) - f \circ \varphi_i^{-1}(x), & x \in \varphi_i(W) \\ 0, & x \in \varphi_i(U_i \backslash W) \end{cases}$$

is C^r. Obviously λ_i is C^r in $\varphi_i(W)$. For the boundary points $\varphi_i(\operatorname{bd} W)$, notice that for $0 \leq k \leq r$, by the definition of N, as $d(y, \varphi_i(U_i \backslash W)) \to 0$, $D^k(\lambda_i)(y) \to 0$ uniformly for $y \in \varphi_i(W)$. Therefore g is indeed C^r.

Finally show that T is continuous. Suppose a topological base $N^r(g, \Phi, \Psi, K, \varepsilon) \subset C^r_s(U,V)$. Take a locally finite open cover $(W_j)_J$ of W and a compact refinement $(W'_j)_J$, then $\Phi' = (\varphi_i, U_i \cap W_j)_{I \times J}$ is still a locally finite family of charts. Then take $K'_{i,j} = K_i \cap W'_j$, $\varepsilon'_{i,j} = \varepsilon_i$ and $\Psi'_{i,j} = (\psi_i, V_i)$, one has $T^{-1}(N^r(g, \Phi, \Psi, K, \varepsilon)) = N^r(g|_W, \Phi', \Psi', K', \varepsilon')$. $\square$

Example 2.28. Suppose $M = \mathbb{R}$ with C^1-structure given by $\{(\mathrm{id}, (-1, \infty)), (f, (-\infty, 1))\}$ with

$$f(x) = \begin{cases} (x + 1/2)^2 - 1/4, & 0 \le x < 1 \\ 1/4 - (1/2 - x)^2, & x < 0 \end{cases}$$

it is not a C^2-atlas because $f \circ \mathrm{id}^{-1}$ on $(-1, 1)$ is not C^2 at 0. To get a C^2-atlas, modify f on $(-1, 1)$: define

$$\widetilde{f}(x) = \begin{cases} 1/4 - (1/2 - x)^2, & x < 0 \\ 2x^3/3 - x^2 + x, & 0 \le x < 1 \end{cases}$$

Now $\{(\mathrm{id}, (-1, \infty)), (\widetilde{f}, (-\infty, 1))\}$ is a C^2-structure that is contained in the previous C^1-structure.

3. De Rham Complex

Definition 3.1. Let U, V be $\mathbb{R}$-vector spaces. Then then a pair $(E, \otimes)$ is called the *tensor product* of U, V if E is an $\mathbb{R}$-vector space and $\otimes$ is a bilinear map from $U \times V$ to E such that for any bilinear map $f : U \times V \to Z$, there is a unique linear map $g : E \to W$ such that $g \circ \otimes = f$, i.e. the following diagram

$$\begin{tikzcd} U \times V \arrow[r, "f"] \arrow[d, "\otimes"'] & Z \\ E \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

commutes. This is also called the *universal property* of tensor products.

Proposition 3.2. *Let U, V be real vector spaces. Suppose there are two tensor products $(E_1, \otimes_1)$, $(E_2, \otimes_2)$ for them, then there is a linear isomorphism such that $f \circ \otimes_1 = \otimes_2$.*

Proof. By the universal property of $\otimes_1$ and $\otimes_2$, one gets the unique linear maps f and g from the following commutative diagram:

$$\begin{tikzcd} E_1 & \\ U \times V \arrow[u, "\otimes_1"'] \arrow[r, "\otimes_2"] \arrow[d, "\otimes_1"] & E_2 \arrow[ul, dashed, "g"'] \\ E_1 \arrow[ur, dashed, "f"'] \arrow[uu, bend left=60, "\mathrm{id}_{E_1}"] & \end{tikzcd}$$

then $g \circ f$ is an unique map from $E_1 \to E_1$, thus $g \circ f = \mathrm{id}_{E_1}$. By interchanging the indices in the above commutative diagram, one also gets $f \circ g = \mathrm{id}_{E_2}$. Therefore f is indeed an isomorphism. $\square$

Remark 3.3. Because of proposition 3.2, one can talk about the tensor product of U and V, say $U \otimes V$.

Proposition 3.4. *Let U, V be real vector spaces, then their tensor product exists.*

Proof. Let T be the real vector space generated by the collection pairs $T_0 = \{(u, v) \mid u \in U, v \in V\}$. Then the elements of T, say t, can be written as the finite sum of elements in T_0, i.e.

$$t = \sum_{i=1}^{n} a_i(u_i, v_i), \quad a_i \in \mathbb{R}, u_i \in U, v_i \in V$$

Put

$$S_1 = \{(au_1 + u_2, v) - a(u_1, v) - (u_2, v) \mid a \in \mathbb{R}, u_1, u_2 \in U, v \in V\}$$
$$S_2 = \{(u, av_1 + v_2) - a(u, v_1) - (u, v_2) \mid a \in \mathbb{R}, u \in U, v_1, v_2 \in V\}$$

and let S be the subspace of T generated by $S_1 \cup S_2$. Let $E = T/S$, and $f : U \times V \to E, (u, v) \to (u, v) + S$. Claim that $(E, \otimes)$ is a tensor product. First observe that

$$f(au_1 + u_2, v) = (au_1 + u_2, v) + S = a(u_1, v) + (u_2, v) + S = af(u_1, v) + f(u_2, v)$$

Similarly $f(u, av_1 + v_2) = af(u, v_1) + f(u, v_2)$. Therefore f is a bilinear map. Now verify the universal property. Suppose $g : U \times V \to Z$ a bilinear map, let $h : E \to Z$ be a map such that $g = h \circ f$. Then h must sends $f(u, v)$ to $g(u, v)$. Since T_0 is a basis of T, $f(u, v)$ spans the quotient space T/S. Therefore h is uniquely determined via linear extension. Observe that

$$h(af(u_1, v) + f(u_2, v)) = h(f(au_1 + u_2, v)) = g(au_1 + u_2, v) = ah(f(u_1, v)) + h(f(u_2, v))$$

and the same argument goes for the case where one fixes u, therefore h is well-defined. $\square$

Definition 3.5. Let V be a real vector space. Then the *wedge product* of V is

$$V \wedge V = V \otimes V / \operatorname{span}\{v \otimes v \mid v \in V\}$$

where the canonical projection is denoted by $\wedge : (u, v) \mapsto u \wedge v$.

Proposition 3.6. *Let V be a real vector space. Then for all skew-symmetric linear maps $f : V \times V \to Z$, there is a unique linear map $g : V \wedge V \to Z$ such that $g \circ \wedge = f$, i.e. the diagram*

$$\begin{tikzcd} V \times V \arrow[r, "f"] \arrow[d, "\wedge"'] & Z \\ V \wedge V \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

commutes.

Proof. This immediately follows from the universal property of quotient spaces and the universal property of tensor products. □

Remark 3.7. Denote $A_k(V)$ to be the collection of skew-symmetric k-linear maps. Define the wedge product on $A_k(V)$ via

$$\wedge : A_k(V) \times A_l(V) \to A_{k+l}(V)$$
$$(a \wedge b)(v_1, \cdots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) a(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) b(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$

There is still some calculations left undone here.

Proposition 3.8. *Let V be a finite dimensional real vector space, then $e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*$ is a basis for $A_k(V)$, where $\{e_1, \cdots, e_n\}$ is a basis for V and $1 \le i_1 \le \cdots \le i_k \le n$.*

Proof. Put $I = (i_1, \cdots i_k)$, write e_I for $(e_{i_1}, \cdots e_{i_k})$ and e_I^* for $e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*$. Then obviously

$$e_I^*(e_J) = \begin{cases} 1, & I = J \\ 0, & I \neq J \end{cases}$$

First show linear independence. Suppose $\sum_I c_I e_I^* = 0$, then by acting e_J on both sides one gets $c_J = 0$. Then show e_I^* spans the whole space. Suppose $f \in A_k(V)$, put $g = \sum_I f(e_I) e_I^*$, then

$$g(e_J) = \sum_I f(e_I) e_I^*(e_J) = f(e_J)$$

therefore g and f agrees on all e_I. By k-linearity and the alternating property, $f = g$. Therefore it is indeed a basis. □

Proposition 3.9. *Let V be a finite dimensional real vector space, then $A_k(V) \cong \bigwedge^k(V^*)$ via $f : \bigwedge^k(V^*) \to A_k(V), e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* \mapsto e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*$.*

Proof. This immediately follows from the fact that f maps the basis of $\bigwedge^k(V^*)$ to the basis of $A_k(V)$ one-to-one. □

Remark 3.10. Let M and N be manifolds, and $f : M \to N$ be a C^∞-map, take charts $(\varphi = (x_1, \cdots, x_m), U)$, $(\psi = (y_1, \cdots, y_n), V)$ of M and N respectively such that $f(U) \subset V$. Then for $p \in U$,

$$\left(T_p f \left(\frac{\partial}{\partial x_1}(p) \right), \cdots, T_p f \left(\frac{\partial}{\partial x_m}(p) \right) \right) = \left(\frac{\partial}{\partial y_1}(f(p)), \cdots, \frac{\partial}{\partial y_n}(f(p)) \right) A(p)$$

where $A(p) = \left[\frac{\partial f_i}{\partial x_j}(p)\right]_{1\leq i\leq n, 1\leq j\leq m}$, $f_i = y_i \circ f : U \to \mathbb{R}$.

Proof. First, note that the tangent vectors $\frac{\partial}{\partial x_j}(p)$ has been identified with derivatives. Therefore

$$(A(p))_{ij} = \frac{\partial f_i}{\partial x_j}(p) = (p) = \frac{d(f_i \circ \varphi^{-1}(te_j + \varphi(p)))}{dt}\Big|_{t=0} = \frac{\partial(\psi \circ f \circ \varphi^{-1})_i}{\partial x_j}(\varphi(p))$$

and

$$\begin{aligned}(T_p f)\left(\frac{\partial}{\partial x_j}(p)\right) &= (T_p f)([p, \varphi, U, e_j]) = [f(p), \psi, V, D(\psi \circ f \circ \varphi^{-1})(\varphi(p))e_j] \\ &= [f(p), \psi, V, \sum_{i=1}^{n} e_i \frac{\partial(\psi \circ f \circ \varphi^{-1})_i}{\partial x_j}(\varphi(p))] \\ &= [f(p), \psi, V, \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j}(p)e_i] = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j}(p)[f(p), \psi, V, e_i] \\ &= \sum_{i=1}^{n} (A(p))_{ij} \frac{\partial}{\partial y_j}(f(p))\end{aligned}$$

□

Definition 3.11. Let M be a manifold. A *differential k-form* on M is a map $w : M \to \bigwedge^k(T^*M) = \sqcup_{p\in M} \bigwedge^k(T_p^*M)$ such that $w(p) \in \bigwedge^k(T_p^*M)$ for all $p \in M$.

Definition 3.12. Let $f \in C^\infty(M, \mathbb{R})$, using the derivative $Tf : TM \to T\mathbb{R}$, one may obtain a 1-form as follows:

$$TM \xrightarrow{Tf} T\mathbb{R} \xrightarrow{\sim} \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

where the last 2 maps are given by $[p, \mathrm{id}, \mathbb{R}, v] \mapsto (p, v) \mapsto v$. The map $df : M \to T^*M$ is called the *differential* of f.

Remark 3.13. Given coordinates $\varphi = (x_1, \cdots, x_m)$, by definition 3.12, one gets 1-forms $dx_1, \cdots dx_m$. Then $(dx_1)_p, \cdots (dx_m)_p$ is the dual basis of $\frac{\partial}{\partial x_1}(p), \cdots, \frac{\partial}{\partial x_m}(p)$.

Proof. Since

$$\begin{aligned}(T_p x_i)(\frac{\partial}{\partial x_j}(p)) &= [x_i(p), \mathrm{id}, \mathbb{R}, D(x_i \circ \varphi^{-1})(\varphi(p))e_j] \\ &= [x_i(p), \mathrm{id}, \mathbb{R}, \frac{\partial(\pi_i \circ \varphi \circ \varphi^{-1})}{\partial x_j}(\varphi(p))] \\ &= [x_i(p), \mathrm{id}, \mathbb{R}, \delta_{ij}]\end{aligned}$$

therefore $(dx_i)_p \left(\frac{\partial}{\partial x_j}(p)\right) = \delta_{ij}$. □

Definition 3.14. Let M be a manifold, then the *cotangent bundle* of M is

$$T^*M = \bigsqcup_{p\in M} T_p^*M = \{(p, f) \mid p \in M, f \in T_p^*M\}$$

Remark 3.15. Give a differential structure on T^*M using charts $(\varphi = (x_1, \cdots, x_m), U)$ of M via $T^*U \to \mathbb{R}^m \times \mathbb{R}^m, (p, \sum_{i=1}^m \lambda_i (dx_i)_p) \mapsto (\varphi(p), \lambda_1, \cdots, \lambda_m)$. It is easy to see that a smooth form must have smooth coefficients under local trivializations. Moreover, define the tensor bundle of type (p,q): $(TM)^{\otimes p} \otimes (T^*M)^{\otimes q}$, and the k-th exterior power of the cotangent bundle on M: $\bigwedge^k (T^*M)$. Now consider the transition functions of the tangent bundle and the cotangent bundle. By the definition of the tangent bundle, suppose coordinate charts $(x_1, \cdots, x_m) = \varphi$ and $(y_1, \cdots, y_m) = \psi$, since $[p, \varphi, U, e_j] = [p, \psi, V, De_j]$ (D is the jocabian),

$$\frac{\partial}{\partial x_j}(p) = D\frac{\partial}{\partial y_i}(p) = \sum_{i=1}^m \frac{\partial y_i}{\partial x_j}\frac{\partial}{\partial y_i}(p)$$

therefore

$$\sum_{j=1}^m v_j \frac{\partial}{\partial x_j}(p) = \sum_{i=1}^m \left(\sum_{j=1}^m v_j \frac{\partial y_i}{\partial x_j}\right)\frac{\partial}{\partial y_i}(p)$$

therefore transition function is $(x, v) \mapsto (\psi \circ \varphi^{-1}(x), Dv)$. In the dual space, suppose $(dx_i)_p = A(dy_j)_p$, then

$$\begin{aligned}\delta_{ij} = (dx_i)_p \left(\frac{\partial}{\partial x_j}(p)\right) &= \sum_k A_{ik}(dy_k)_p \left(\sum_l D_{jl}\left(\frac{\partial}{\partial y_l}(p)\right)\right)\\ = \sum_{k,l} A_{ik} D_{jl}\delta_{kl} &= \sum_k A_{ik} D_{jk}\end{aligned}$$

therefore $AD^T = I$, $A = (D^T)^{-1}$. By the same argument the transition function is $(x, v) \mapsto (\psi \circ \varphi^{-1}(x), Av)$.

Definition 3.16. A differential k-form w is said to be *smooth* if $w \in C^\infty(M, \bigwedge^k(M))$.

Remark 3.17. Write $\Omega^k(M)$ for the collection of all smooth differential k-forms for $k = 1, 2, \cdots$. For $k = 0$, $\Omega^0(M) = C^\infty(M, \mathbb{R})$. They can be seen as $\mathbb{R}$-vector spaces.

Example 3.18. (a) Find $w \in \Omega^1(\mathbb{R}^2 \backslash \{0\})$ which satisfies

$$\begin{cases} w\left(-y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}\right) = 1, \\ w\left(x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}\right) = 0 \end{cases}$$

To find such w, suppose anstaz $w = adx + bdy$. Then

$$\begin{cases} -ay + bx = 1, \\ ax + by = 0 \end{cases} \iff \begin{cases} a = \dfrac{-y}{x^2+y^2}, \\ b = \dfrac{x}{x^2+y^2} \end{cases}$$

therefore $w = \dfrac{xdy - ydx}{x^2+y^2}$.

(b) There is a map $d : \Omega^0(\mathbb{R}^2 \backslash \{0\}) \to \Omega^1(\mathbb{R}^2 \backslash \{0\}), f \mapsto df$. Claim that $w \notin \operatorname{im}(d)$. For assume on contrary there exists f such that $w = df$. Then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{xdy - ydx}{x^2+y^2}$$

since dx, dy is a basis for each $p \in M$, $\dfrac{\partial f}{\partial x} = \dfrac{-y}{x^2+y^2}$ and $\dfrac{\partial f}{\partial y} = \dfrac{x}{x^2+y^2}$. Put $g(t) = f(c(t))$ where $c(t) = (\cos t, \sin t)$, then

$$\int_0^{2\pi} g'(t)dt = \int_0^{2\pi} \nabla f(c(t)) \cdot (c'(t))^T dt = \int_0^{2\pi} 1dt = 2\pi$$

however by the fundamental theorem of calculus,

$$\int_0^{2\pi} g'(t)dt = g(2\pi) - g(0) = f(0) - f(0) = 0$$

contradiction.

(c) There is another map $d : \Omega^1(\mathbb{R}^2\backslash\{0\}) \to \Omega^2(\mathbb{R}^2\backslash\{0\})$ given by $d(adx + bdy) = da \wedge dx + db \wedge dy$, so one has a chain

$$\Omega^0(\mathbb{R}^2\backslash\{0\}) \xrightarrow{d_1} \Omega^1(\mathbb{R}^2\backslash\{0\}) \xrightarrow{d_2} \Omega^2(\mathbb{R}^2\backslash\{0\})$$

this is called the *de Rham complex* of $\mathbb{R}^2\backslash 0$. Since

$$d(df) = \frac{\partial^2 f}{\partial x \partial y} dx \wedge dy + \frac{\partial^2 f}{\partial y \partial x} dy \wedge dx = 0$$

it is indeed a cochain complex. Also, since $dw = 0$, $H^1_{dR}(\mathbb{R}^2\backslash\{0\}) = \ker d_1 / \operatorname{im} d_0$ is non-trivial. Claim that $H^1_{dR}(\mathbb{R}^2\backslash\{0\}) \cong \mathbb{R}$ via

$$\Phi : [w] \mapsto \frac{1}{2\pi} \int_0^{2\pi} w(c'(t))dt$$

By (b), since $1 \in \operatorname{im}\Phi$, Φ is surjective. For the injectivity, one needs to show that if a form $w \in \Omega^1(\mathbb{R}\backslash\{0\})$ is closed, and $\int_0^{2\pi} w(c'(t))dt = 0$, then it is exact. Assume $w = fdx + gdy$. Since w is closed, $dw = 0$, therefore $\frac{\partial f}{\partial y}(p) = \frac{\partial g}{\partial x}(p)$ for all $p \in \mathbb{R}^2\backslash\{0\}$, the integral over the boundary of a simply connected domain is always 0 by Green's theorem. To emphasize that the vector line integrals does not depend on the parameterization, denote $\int_C w$ the integration of w counterclockwise along the curve C. Now for all $p \in \mathbb{R}^2\backslash\{0\}$, put C_p a curve that starts at $(1, 0)$ and ends at $p = (r\cos\theta, r\sin\theta)$ where $r > 0$ and $\theta \in [0, 2\pi)$ as follows: it first moves from $(1, 0)$ to $(\cos\theta, \sin\theta)$ counterclockwise along the unit circle, then goes to p by the straitline connecting the origin, p and $(\cos\theta, \sin\theta)$. Put $a(p) = \int_{C_p} w$ and claim that $w = da$. For $p \in \mathbb{R}^2\backslash 0$ that not on the positive x-axis, say $p = (x_0, y_0)$, define C_h to be the line that starts at p and ends at $(x_0 + h, y_0)$. Since p is not on the positive x-axis, h can always be taken so that C_h and C_{-h} does not intersect the positive x-axis. Then

$$\frac{\partial a}{\partial x}(p) = \lim_{h\to 0} \frac{a(p + (h, 0)) - a(p)}{h} = \lim_{h\to 0} \frac{\int_{C_h} w}{h} = \lim_{h\to 0} \frac{\int_0^h f(x_0 + x)dx}{h} = f(p)$$

by the same argument one can show that $\frac{\partial a}{\partial y} = g$. If p is on the positive x-axis, then by the same argument the right derivative with respect to y is still $g(p)$. For the left derivative, for a point $(x_0, h) = r(\cos\theta, \sin\theta)$, take another path C'_p from $(1, 0)$ clockwise along the unit circle to $(\cos\theta, \sin\theta)$, then again take the straitline to (x_0, h). Because the integration of w along the unit circle is zero, $\int_{C_p} w = \int_{C'_p} w$. Using this, one finds

$$\lim_{h\to 0^-} \frac{a(p + (0, h)) - a(p)}{h} = \lim_{h\to 0^-} \frac{\int_{C'_{p+(0,h)}} w - \int_{C'_p} w}{h} = \lim_{h\to 0^-} \frac{\int_{C_h} w}{h} = g(p)$$

therefore the derivative with respect to y is still well-defined and agrees with g on the positive x-axis. Therefore $w = df$. For $w = a(x, y)dx \wedge dy \in \Omega^2(\mathbb{R}^2\backslash\{0\})$, take $f = \int_0^x a(x, y)dx$, then $d(fdy) = w$, hence $H^2_{dR}(\mathbb{R}^2\backslash\{0\}) \cong 0$.

(d) Find $\Omega^1(S^1)$. Consider S^1 as a submanifold of $\mathbb{R}^2$, then $T_pS^1 = \left\{ a\dfrac{\partial}{\partial x}(p) + b\dfrac{\partial}{\partial y}(p) \mid ax_p + by_p = 0 \right\}$. Therefore $\Omega^1(S^1) = C^\infty(S^1, \mathbb{R})(-ydx + xdy)|_{TS^1}$. Although all the differential forms on S^1 comes from the restrictions of differential forms on $\mathbb{R}^2$, this is not true in general for submanifolds. For example, $(0,1)$ is a submanifold of $\mathbb{R}$, and $\sin(1/x)dx$ is a differential form that cannot be extended to a form on $\mathbb{R}$.

Definition 3.19. A quadruple $(A, + : A \times A \to A, \cdot : \mathbb{R} \times A \to A, \odot : A \times A \to A)$ is called an $\mathbb{R}$-*algebra* if
(a) $(A, +, \cdot)$ is an $\mathbb{R}$-vector space;
(b) $(A, +, \odot)$ is a ring;
(c) $\forall r_1, r_2 \in \mathbb{R}$, and $\forall a_1, a_2 \in A$, $(r_1, a_1) \odot (r_2, a_2) = (r_1 r_2)(a_1 \odot a_2)$.

Definition 3.20. Let A be an $\mathbb{R}$-algebra, and let $A_i \leq A$ be sub-vector spaces for $i \in \mathbb{N}_0$, then A is called a *graded algebra* with *grading* $(A_i)_{\mathbb{N}_0}$.

Definition 3.21. Let A be an $\mathbb{R}$-algebra. An $\mathbb{R}$-linear map $D : A \to A$ is called a *derivation* of A if it satisfies the leibniz rule, i.e. $D(ab) = D(a)b + aD(b)$. An $\mathbb{R}$-linear map $D : A \to A$ is called an *anti-derivation* of A if it satisfies the leibniz rule for anti-derivations, i.e. for all $a \in A_i, b \in A$, $D(ab) = D(a)b + (-1)^i aD(b)$.

Example 3.22. (a) $\Omega^*(M) = \bigoplus_{i=0}^\infty \Omega^i(M)$ with $\wedge$ is a graded algebra.
(b) $D : \Omega^*(\mathbb{R}^n) \to \Omega^*(\mathbb{R}^n)$ defined via $D(\sum_I a_I dx^I) = \sum_I da_I \wedge dx^I$ where $I = (i_1, \cdots i_l)$ for $1 \leq i_1 < \cdots < i_l \leq n$ and $dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_l}$ is an anti-derivation that satisfies $D|_{\Omega^0(\mathbb{R}^n)} = d$ and $D \circ D = 0$.

Proof. Since the wedge product and the differential $d : C^\infty(\mathbb{R}^m) \to \Omega^1(\mathbb{R}^n)$ is $\mathbb{R}$-linear, D is $\mathbb{R}$-linear. Now check the leibniz rule of anti-derivations for D. One only needs to consider the elements of the form adx^I because of the additivity of D. Then

$$\begin{aligned} D(adx^I \wedge bdx^J) &= D(ab \cdot \operatorname{sgn}(\rho) dx^{I \cup J}) = \operatorname{sgn}(\rho) d(ab) \wedge dx^{I \cup J} \\ &= d(ab)dx^I \wedge dx^J = ((da)b + adb) \wedge dx^I \wedge dx^J \\ &= (da \wedge dx^I) \wedge (bdx^J) + (-1)^{|I|} adx^I \wedge db \wedge dx^J \\ &= D(adx^I) \wedge (bdx^J) + (-1)^{|I|} adx^I \wedge D(bdx^J) \end{aligned}$$

thus D is indeed an anti-derivative. Obviously $D|_{\Omega^0(\mathbb{R}^n)} = d$ by definition, and since

$$\begin{aligned} D(D(adx^I)) = D(da \wedge dx^I) &= D\left(\sum_{i=1}^n \frac{\partial a}{\partial x_i} dx_i \wedge dx^I \right) \\ &= \sum_{i=1}^n d\left(\frac{\partial a}{\partial x_i} \right) \wedge dx_i \wedge dx^I \\ &= \sum_{j=1}^n \sum_{i=1}^n \frac{\partial^2 a}{\partial x_i \partial x_j} dx_j \wedge dx_i \wedge dx^I = 0 \end{aligned}$$

by linearity $D \circ D = 0$. □

Definition 3.23. Let $f \in C^\infty(M, N)$ and $w \in \Omega^k(N)$. Define $f^*w \in \Omega^k(M)$ via

$$(f^*w)_p(v_1, \cdots, v_k) = w_{f(p)}((T_pf)v_1, \cdots, (T_pf)v_k)$$

it is called the *pullback* of w under f. $f^* : \Omega^*(N) \to \Omega^*(M)$ is an $\mathbb{R}$-algebra homomorphism.

Remark 3.24. Obviously there is $(f \circ g)^* = g^* \circ f^*$, and if f is a diffeomorphism, then f^* is an isomorphism. Also, f^* has the local property, i.e. if $w_1 = w_2$ around $f(p)$, then $f^*w_1 = f^*w_2$ around p.

Definition 3.25. An *exterior derivative* on a manifold M is an $\mathbb{R}$-linear map $D : \Omega^*(M) \to \Omega^*(M)$ which satisfies:

(a) D is an anti-derivation;
(b) $D \circ D = 0$;
(c) $D|_{\Omega^0(M)} = d$.

Theorem 3.26. *Let M be a manifold, then there exists a unique exterior derivative on M.*

Proof. First show uniqueness. Let D and D' be exterior derivatives on M. Take a chart (φ, U) on M and $p \in U$, then for $w \in \Omega^*(M)$, since $D|_U = D'_U$ by lemma 3.29,

$$D(w)(p) = D|_U(w|_U)(p) = D'_U(w|_U) = D'(w)(p)$$

therefore $D = D'$. For the existence, for every chart (φ, U), there is a unique exterior derivative $D_{\varphi(U)}$: $\Omega^*(\varphi(U)) \to \Omega^*(\varphi(U))$. Since φ^* is an isomorphism, it uniquely induces an exterior derivative $D_U : \Omega^*(U) \to \Omega^*(U)$. Define $D : \Omega^*(M) \to \Omega^*(M)$ via $D(w)(p) = D_U(w|_U)(p)$. Since D_U is an exterior derivative, D is also an exterior derivative. $\square$

Lemma 3.27. *Suppose U is an open set of $\mathbb{R}^m$, then it has a unique exterior derivative.*

Proof. The existence follows from example 3.22 (b). For the uniqueness, let D' be another exterior derivative. If $w \in \Omega^0(U)$, then by definition 3.25 (3), $D(w) = D'(w) = dw$. Now assume D agrees with D' for all $w \in \bigoplus_{i=0}^k \Omega^i(M)$, take $adx^I \in \Omega^{k+1}(M)$. Then

$$\begin{aligned} D'(adx^I) &= D'(a) \wedge dx^I + (-1)^0 aD'(dx^I) = da \wedge dx^I + aD'(dx^I) \\ &= D(adx^I) + aD'(dx^I) \end{aligned}$$

since

$$D'(dx^I) = D'(D(x_{i_1} dx^{I\setminus\{i_1\}})) = D'(D'(x_{i_1} dx^{I\setminus\{i_1\}})) = 0$$

therefore $D = D'$ by induction. $\square$

Lemma 3.28. *Suppose M a manifold, $p \in M$, and D an exterior derivative on M. If there exists an open set U of M with $p \in U$ such that $w|_U = 0$, then there exists an open neighbourhood U' of p such that $D(w)|_{U'} = 0$.*

Proof. W.l.o.g. assume U is part of a chart (φ, U). Take $\lambda : M \to [0,1]$ a C^∞-map such that $\lambda = 0$ on U' and $\lambda = 1$ on $M \backslash U$. (Just take U' small enough so $\overline{U'} \subset U$, and take a bump function b such that $b = 1$ on U' and $b = 0$ on $M \backslash U$, then $\lambda = 1 - b$). Then

$$D(w)|_{U'} = D(\lambda w)|_{U'} = d\lambda \wedge w|_{U'} + \lambda \wedge D(w)|_{U'}$$

since $w|_{U'} = 0$ and $\lambda_{U'} = 0$, $D(w)|_{U'} = 0$. $\square$

Lemma 3.29. *Suppose (φ, U) a chart of M. Then an exterior derivative D on M uniquely induces an exterior derivative D_U on U.*

Proof. Then define $D_U : \Omega^*(U) \to \Omega^*(U)$ as follows: for $w \in \Omega^*(U)$, $D_U(w)(p) = D(\lambda w)(p)$, where $\lambda = 1$ around p and $\lambda = 0$ on $M \backslash V$ such that $p \in V \subset \overline{V} \subset U$. This is well-defined by lemma 3.28. D_U is a exterior derivative on $\Omega^*(U)$, since:

(1) $D_U \circ D_U(w)(p) = D_U(\lambda D_u(w))(p) = D(\lambda D(\lambda w))(p) = DD(\lambda w))(p) = 0$;
(2) Take $w \in \Omega^0(U)$, then $D_U(w)(p) = D(\lambda w)(p) = d(\lambda w)(p) = d_U(w)(p)$;

(3) Suppose two bump functions λ_1, λ_2 around p, then $\lambda = \lambda_1\lambda_2$ is still a bump function around p. Then

$$\begin{aligned} D_U(a \wedge b)(p) &= D(\lambda a \wedge b)(p) = D((\lambda_1 a) \wedge (\lambda_2 b))(p) \\ &= (D(\lambda_1 a) \wedge \lambda_1 b + (-1)^{\deg a} \lambda_1 a \wedge D(\lambda_2 b))(p) \\ &= (D_U(a) \wedge b + (-1)^{\deg a} a \wedge D_U(b))(p) \end{aligned}$$

Therefore a exterior derivative D on M. Since U is diffeomorphic to $\varphi(U)$, the following diagram commutes:

$$\begin{tikzcd} \Omega^*(U) \arrow[r, "D_U"] & \Omega^*(U) \\ \Omega^*(\varphi(U)) \arrow[u, "\varphi^*"] \arrow[r, dashed, "D_{\varphi(U)}"'] & \Omega^*(\varphi(U)) \arrow[u, "\varphi^*"] \end{tikzcd}$$

since $D_{\varphi(U)}$ is unique by lemma 3.27, D_U is also unique. $\square$

Definition 3.30. Let M be a manifold. The complex

$$0 \xrightarrow{d^{(-1)}} \Omega^0(M) \xrightarrow{d^{(0)}} \Omega^1(M) \longrightarrow \cdots \longrightarrow \Omega^{m-1}(M) \xrightarrow{d^{(m)}} \Omega^m(M) \longrightarrow 0$$

is called the *de Rham complex* of M, where $d^{(i)} = D|_{\Omega^i(M)}$. Also define $Z^i(M) = \ker(d^{(i)})$ the set of *cocycles* or *closed forms*, and $B^i(M) = \operatorname{im} d^{(i-1)}$ the set of *coboundaries* or *exact forms*. $H^i_{dR}(M) = Z^i(M)/B^i(M)$ is called the *i-th de Rham cohomology* of M.

Example 3.31. Let M be a manifold with connected components M_i for $i \in I$, I is countable. Then $H^0_{dR}(M) \cong \mathbb{R}^{|I|}$, because a function $f \in C^\infty(M, \mathbb{R})$ with $df = 0$ is constant on every M_i.

Example 3.32. Consider the de Rham cohomology of S^1:

$$0 \longrightarrow \Omega^0(S^1) \xrightarrow{d^{(0)}} \Omega^1(S^1) \longrightarrow 0$$

Since S^1 is connected, $H^0_{dR}(S^1) \cong \mathbb{R}$. Consider $\Phi : \Omega^1(S^1) = Z^1(S^1) \to \mathbb{R}$, with

$$\Phi(w) = \int_0^{2\pi} w_{c(t)}(c'(t))dt$$

where

$$c'(t) = (T_t c)\left(\frac{\partial}{\partial t}\right) = c_1'(t)\frac{\partial}{\partial x}(c(t)) + c_2'(t)\frac{\partial}{\partial y}(c(t))$$

also, $\operatorname{im}(d^{(0)}) \subset \ker \Phi$ since

$$\begin{aligned} \int_0^{2\pi} df_{c(t)}(c'(t))dt = \int_0^{2\pi} d(f \circ c)_t \left(\frac{\partial}{\partial t}(t)\right) dt &= \int_0^{2\pi} (f \circ c)'(t)dt \\ = (f \circ c)(2\pi) - (f \circ c)(0) = 0 & \end{aligned}$$

and Φ is surjective since

$$\begin{aligned} \Phi(-ydx + xdy|_{TS^1}) &= \int_0^{2\pi} -c_2(t)dx_{c(t)}(c'(t)) + c_1(t)dy_{c(t)}(c'(t))dt \\ &= \int_0^{2\pi} -c_2(t)c_1'(t) + c_1(t)c_2'(t)dt = 2\pi \end{aligned}$$

Now show $\ker\Phi \subset \operatorname{im}(d^{(0)})$: let $w \in \ker\Phi$, put $f(c(t)) = \int_0^t w_{c(s)}(c'(s))ds$, f is well-defined because $w \in \ker\Phi$. Then

$$\begin{aligned}(df)_{c(t)}(c'(t)) &= (df_{c(t)} \circ T_t c)\left(\frac{\partial}{\partial t}(t)\right) = d(f \circ c)_t\left(\frac{\partial}{\partial t}(t)\right) \\ &= (f \circ c)'(t) = \left(\int_0^t w_{c(s)}c'(s)ds\right)' = w_{c(t)}(c'(t))\end{aligned}$$

since $c'(t)$ is non-zero everywhere on S^1, and $T_{c(t)}S^1$ is of dimension 1, $df = w$. Thus $\ker\Phi = \operatorname{im}(d^{(0)})$. Therefore

$$\mathbb{R} \cong Z^1(S^1)/\ker\Phi = Z^1(S^1)/B^1(S^1) = H^1_{dR}(S^1)$$

now all the cohomology group of S^1 is known.

Example 3.33 (Poincaré lemma)**.** Suppose $n \in \mathbb{N}^0$, then

$$H^k_{dR}(\mathbb{R}^n) = \begin{cases}\mathbb{R}, & k = 0\\ 0, & k \geq 1\end{cases}$$

Proof. Since $\mathbb{R}^n$ is connected, $H^0_{dR}(\mathbb{R}^n) \cong \mathbb{R}$. For $k \geq 1$, take $w = adx_{i_1} \wedge \cdots \wedge dx_{i_k}$ such that $dw = 0$, i.e.

$$\sum_{j \notin \{i_1, \cdots, i_k\}} \frac{\partial a}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0$$

thus for all $j \notin \{i_1, \cdots i_k\}$, $\dfrac{\partial a}{\partial x_j} = 0$, thus a only relies on $x_{i_1}, \cdots, x_{i_k}$. Define

$$f(x_{i_1}, \cdots, x_{i_k}) = \int_0^{x_{i_1}} a(x_{i_1}, \cdots, x_{i_k})ds$$

then $d(fdx_{i_2} \wedge \cdots \wedge dx_{i_k}) = a(x_{i_1}, \cdots, x_{i_k})dx_{i_1} \wedge \cdots \wedge x_{i_k}$. □

Remark 3.34. There are usually four tools for computing the de Rham cohomology:
(a) the Poincaré lemma;
(b) integration on manifolds;
(c) Mayer-Vietoris sequence;
(d) homotopy invariance.

Lemma 3.35. *Let $f \in C^\infty(M, N)$. Then the exterior derivatives d_M and d_N satisfy $d_M f^* = f^* d_N$, i.e. the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^0(M) & \xrightarrow{d_M} & \Omega^1(M) & \xrightarrow{d_M} & \Omega^2(M) & \longrightarrow & \cdots \\ & & {\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow & & \\ 0 & \longrightarrow & \Omega^0(N) & \xrightarrow{d_N} & \Omega^1(N) & \xrightarrow{d_N} & \Omega^2(N) & \longrightarrow & \cdots \end{array}$$

commutes.

Proof. For all $w \in \Omega^*(N)$, take any $f(p) \in N$, it can be written as linear combinations of $adx_1 \wedge \cdots \wedge dx_s$ locally around $f(p)$ (by taking charts). W.l.o.g assume $w = adx_1 \wedge \cdots \wedge dx_s$, then the form $w' = ad(\lambda x_1) \wedge \cdots \wedge d(\lambda x_s)$ agrees with w locally around $f(p)$. Therefore $f^*w = f^*w'$ around p by the local property of f. Since f^* is an $\mathbb{R}$-algebra homomorphism, one only needs to look at the differential forms in $\Omega^0(N) \cup B^1(N)$.

Suppose $g \in \Omega^0(N)$, $p \in M$, $v = \sum_{i=1}^m v_i \dfrac{\partial}{\partial x_i} \in T_p(M)$, then

$$\begin{aligned} f^*(d_N g)_p(v) = (d_N g)(T_p f v) &= \left(\sum_{k=1}^n \frac{\partial g}{\partial y_k} dy_k\right)\left(\sum_{i=1}^n \sum_{j=1}^m \frac{\partial(\psi \circ f \circ \varphi^{-1})}{\partial x_j} v_j \frac{\partial}{\partial y_i}\right) \\ &= \sum_{i=1}^n \sum_{j=1}^m \frac{\partial g}{\partial y_i} \frac{\partial(\psi \circ f \circ \varphi^{-1})}{\partial x_j} v_j = \sum_{j=1}^m \frac{\partial(g \circ f \circ \varphi^{-1})}{\partial x_j} v_j \\ &= d(g \circ f)_p(v) = d_M(f^*(g))_p(v) \end{aligned}$$

therefore $f^* d_N g = d_M f^* g$. For the case where $d_N g \in B^1(N)$, $p \in M$, $v \in T_p(M)$,

$$f^*(d_N d_N g) = f^*(0) = 0, \quad d_M f^*(d_N g) = d_M d_M f^*(g) = 0$$

where the second equality used the result for $g \in \Omega^0(N)$. $\square$

Remark 3.36. Suppose $f \in C^\infty(M, N)$, then f^* induces a map $H^k_{dR}(N) \to H^k_{dR}(M)$ for $k = 0, 1, 2, \cdots$.

Proof. By lemma 3.35, for $k \geq 0$, one has $f^*(\operatorname{im} d_N^{(i)}) \subset \operatorname{im} d_M^{(i)}$. Suppose w such that $d_N^{(i+1)} w = 0$, then $d_M^{(i+1)}(f^* w) = 0$, therefore $f^*(\ker d_N^{(i+1)}) \subset \ker d_M^{(i+1)}$. $\square$

Lemma 3.37. *Let M be a manifold. Consider the two maps $f_i : M \to M \times [0,1]$ for $i = 0, 1$ with $f_i(p) = (p, i)$. Then there exists a linear map $L : \Omega^*(M \times [0,1]) \to \Omega^*(M)$ such that $f_1^* - f_0^* = d_M \circ L + L \circ d_{M \times [0,1]}$, i.e. the following non-commutative diagram:*

$$\begin{tikzcd} \cdots \arrow[r] & \Omega^{k-1}(M \times [0,1]) \arrow[r] & \Omega^k(M \times [0,1]) \arrow[r] \arrow[dl, "L^{(k)}"'] & \Omega^{k+1}(M \times [0,1]) \arrow[r] \arrow[dl, "L^{(k+1)}"'] & \cdots \\ \cdots \arrow[r] & \Omega^{k-1}(M) \arrow[r] & \Omega^k(M) \arrow[r] & \Omega^{k+1}(M) \arrow[r] & \cdots \end{tikzcd}$$

Proof. Suppose $N = M \times [0,1]$. Then there are maps: $\pi_1 : N \to M, (p,s) \mapsto p$, $\pi_2 : N \to [0,1], (p,s) \mapsto s$, and $\iota : M \to N, p \mapsto (p,1)$. Since $\pi \circ \iota = \mathrm{id}_M$, $\pi^* : \Omega^*(M) \to \Omega^*(N)$ is injective. Also, one has an isomorphism $\Phi_{(p,s)} = T_{(p,s)}\pi_1 \oplus T_{(p,s)}\pi_2 : T_{(p,s)}N \to T_pM \oplus T_s[0,1]$. Take $v \in T_pM$, define vector fields $X_v : [0,1] \to TN$ via $X_v(s) = \Phi^{-1}_{(p,s)}(v, 0)$ and

$$\frac{\partial}{\partial t}(p,s) = \Phi^{-1}_{(p,s)}\left(0, \frac{\partial}{\partial t}(s)\right)$$

Then define L. For $w \in \Omega^k(N)$, if $k = 0$, then $L(w) = 0$. Otherwise put

$$L(w)_p(v_1, \cdots, v_{k-1}) = \int_0^1 w_{(p,s)}\left(\frac{\partial}{\partial t}(p,s), X_{v_1}(s), \cdots, X_{v_k}(s)\right) ds$$

By the definition of L, it is obvious that it has the local property, i.e. if there exists $U \subset M$ an open subset and $w_1, w_2 \in \Omega^*(N)$ such that $w_1|_{U \times [0,1]} = w_2|_{U \times [0,1]}$, then $L(w_1)|_U = L(w_2)|_U$. Then by the additivity of L, one only needs to consider the case where $w = a dx^I$ and $w = a dt \wedge dx^I$ with $a \in C^\infty(U \times [0,1], \mathbb{R})$ compact supported (otherwise just multiply a bump function).

(a) Suppose $w = a \in C_c^\infty(U \times [0,1], \mathbb{R})$, then

$$(f_1^*(a) - f_0^*(a))(p) = (a \circ f_1)(p) - (a \circ f_0)(p) = a(p,1) - a(p,0)$$

$$(d_M \circ L)(a) = d_M(L(a)) = d_M 0 = 0$$

$$(L\circ d_N)(a) = L\left(\sum_{i=1}^m \frac{\partial a}{\partial x_i}dx_i + \frac{\partial a}{\partial t}dt\right)(p) = \int_0^1 \frac{\partial a}{\partial t}(p,s)ds = a(p,1)-a(p,0)$$

this verifies case (a).

(b) Suppose $w = adx^I$ where $a \in C_c^\infty(U\times[0,1],\mathbb{R})$ and $1 \le |I| = k \le m$. In this case, since $(dx_j)_{(p,s)}(\frac{\partial}{\partial t}(p,s)) = 0$, $L(adx^I) = 0$. therefore

$$L\circ d_N(adx^I) = L\left(\frac{\partial a}{\partial t}dt\wedge dx^I\right)$$

Then take $v_1,\cdots,v_{k-1}\in T_pM$,

$$\begin{aligned} L\left(\frac{\partial a}{\partial t}dt\wedge dx^I\right)_p(v_1,\cdots,v_{k-1}) &= \int_0^1 \frac{\partial a}{\partial t}(p,s)(dx^I)_{(p,s)}(X_{v_1}(s),\cdots,X_{v_{k-1}}(s))ds \\ &= \left(\int_0^1 \frac{\partial a}{\partial t}(p,s)ds\right)(dx^I)_p(v_1,\cdots,v_{k-1}) \\ &= (f_1^*(a)(p)-f_0^*(a)(p))(dx^I)_p(v_1,\cdots,v_{k-1}) \\ &= (f_1^*(adx^I)-f_0^*(adx^I))_p(v_1,\cdots,v_k) \end{aligned}$$

this concludes case (b).

(c) Suppose $w = adt\wedge dx^I$, $a\in C_c^\infty(U\times[0,1],\mathbb{R})$, since $f_i^*(dt)=0$,

$$f_1^*(w)-f_0^*(w) = f_1^*(adt)\wedge f_1^*(dx^I) - f_0^*(adt)\wedge f_0^*(dx^I) = 0$$

one also has

$$\begin{aligned} (d_M\circ L)(w)_p(v_1,\cdots,v_{k-1}) &= d_M\left(\left(\int_0^1 a(\bullet,s)ds\right)(dx^I_\bullet)\right) \\ &= d_M\left(\int_0^1 a(\bullet,s)ds\right)_p\wedge(dx^I)_p + \int_0^1 a(p,s)ds\left(d_M(dx^I)\right)_p \\ &= d_M\left(\int_0^1 a(\bullet,s)ds\right)_p\wedge(dx^I)_p \end{aligned}$$

and

$$\begin{aligned} (L\circ d_N)(w)_p &= L(da\wedge dt\wedge dx^I)_p \\ &= L\left(\sum_{i=1}^m \frac{\partial a}{\partial x_j}dx_j\wedge dt\wedge dx^I\right)_p = -L\left(\sum_{i=1}^m \frac{\partial a}{\partial x_j}dt\wedge dx_j\wedge dx^I\right)_p \\ &= -\left(\sum_{i=1}^m\int_0^1 \frac{\partial a}{\partial x_j}(p,s)ds\cdot dx_j\right)\wedge dx^I \\ &= -d_M\left(\int_0^1 a(\bullet,s)ds\right)_p\wedge dx^I \end{aligned}$$

this concludes case (c), the proof is done. □

Theorem 3.38. *Suppose $f,g\in C^\infty(M,N)$ are smoothly homotopic, i.e. there exists $H\in C^\infty(M\times[0,1],N)$ such that $H(\bullet,0)=f(\bullet)$ and $H(\bullet,1)=g(\bullet)$, then $f^*=g^*$ on $H^k_{dR}(N)$ for $k=0,1,2,\cdots$.*

Proof. Take maps $f_i : M \times [0,1] \to M$ and L from lemma 3.37. Then $H \circ f_0 = f$, $H \circ f_1 = g$. Therefore

$$g^* - f^* = (f_1^* - f_0^*) \circ H^* = d_M \circ L \circ H^* + L \circ d_{M\times[0,1]} \circ H^* = d_M \circ L \circ H^* + L \circ H^* \circ d_N$$

suppose $w \in Z^k(N)$, then $(g^* - f^*)(w) = d_M \circ L \circ H^*(w) \in B^k(M)$, hence the difference is zero on the cohomology groups. □

Theorem 3.39. *Suppose M and N are homotopic manifolds without boundary, i.e. there exists $f \in C^\infty(M,N)$ and $g \in C^\infty(N,M)$ such that $f \circ g$ and $g \circ f$ are smoothly homotopic to id_M, id_N respectively, then $H^k_{dR}(N) \cong H^k_{dR}(M)$ for $k = 0, 1, 2, \cdots$.*

Proof. By theorem 3.38, $g^* \circ f^* = (f \circ g)^* = (id_M)^* = id_{H^k_{dR}(M)}$, and $f^* \circ g^* = (id_N)^* = id_{H^k_{dR}(N)}$ on all the respective cohomology groups. Therefore f^* is the isomorphism required. □

Example 3.40. Suppose $n \in \mathbb{N}^+$. Then S^n and $\mathbb{R}^{n+1}\backslash\{0\}$ are homotopic via the following maps:

$$S^n \xrightarrow{\iota} \mathbb{R}^{n+1}\backslash\{0\} \xrightarrow{\pi} S^n$$

where ι is the inclusion map and π is given by $x \mapsto \dfrac{x}{|x|_2}$. Then $\pi\circ\iota = \mathrm{id}_{S^n}$, and $\iota\circ\pi$ is homotopic to $\mathrm{id}_{\mathbb{R}^{n+1}\backslash\{0\}}$ via $H : (\mathbb{R}^{n+1}\backslash\{0\}) \times [0,1] \to \mathbb{R}^{n+1}\backslash\{0\}$, $H(x,t) = tx + (1-t)\dfrac{x}{|x|_2}$, it is obvious that $H(x,1) = \mathrm{id}_{\mathbb{R}^{n+1}\backslash\{0\}}(x)$ and $H(x,0) = \iota \circ \pi(x)$. Therefore $H^k_{dR}(S^n) \cong H^k_{dR}(\mathbb{R}^{n+1}\backslash\{0\})$ for all k.

Definition 3.41. Let M be a manifold. Denote $H^*_{dR}(M) = \bigoplus_{i=0}^\infty H^i_{dR}(M)$, then it is an $\mathbb{R}$-algebra with respect to the following product:

$$\smile : H^*_{dR}(M) \times H^*_{dR}(M) \to H^*_{dR}(M), \quad [w_1] \smile [w_2] = [w_1 \wedge w_2]$$

this map is called the *cup product*.

Proposition 3.42. *The cup product for M is well-defined.*

Proof. Take $[w_1] \in H^k_{dR}(M)$, $[w_2] \in H^l_{dR}(M)$ and $\rho \in \Omega^{k-1}(M)$, then $[w_1 + d\rho] = [w_1]$. To verify it is well-defined, one has to show $[(w_1 + d\rho) \wedge w_2] = [w_1 \wedge w_2]$, i.e. $d\rho \wedge w_2$ is exact. Since

$$d(\rho \wedge w_2) = d\rho \wedge w_2 + (-1)^{k-1}\rho \wedge dw_2 = d\rho \wedge w_2$$

the proof is done. □

Example 3.43. $H^*_{dR}(S^1) \cong \mathbb{R} \oplus \mathbb{R}$ as real vector spaces, and $(H^*_{dR}(S^1), \smile) \cong \mathbb{R}[x]/\langle x^2 \rangle$ as $\mathbb{R}$-algebras.

Theorem 3.44. *Let M be a manifold, and U, V be open subsets of M such that $M = U \cup V$. Denote the inclusion maps $\iota_{U,M} : U \to M$, $\iota_{V,M} : V \to M$ and similarly $\iota_{U\cap V,U}$, $\iota_{U\cap V,V}$. Then:*

(a) the sequence of $\mathbb{R}$-vector spaces

$$0 \longrightarrow \Omega^*(U \cup V) \xrightarrow{\alpha} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\beta} \Omega^*(U \cap V) \longrightarrow 0$$

is exact, where $\alpha = \iota^*_{U,M} + \iota^*_{V,M}$ and $\beta = \iota^*_{U\cap V,U} - \iota^*_{U\cap V,V}$.

(b) there is the following long exact sequence (Mayer-Vietoris sequence)

$$\begin{tikzcd} 0 \arrow[r] & H^0_{dR}(U \cup V) \arrow[r, "\alpha"] & H^0_{dR}(U) \oplus H^0_{dR}(V) \arrow[r, "\beta"] & H^0_{dR}(U \cap V) \arrow[dll, "\delta"] & \\ & H^1_{dR}(U \cup V) \arrow[r, "\alpha"] & H^1_{dR}(U) \oplus H^1_{dR}(V) \arrow[r, "\beta"] & H^1_{dR}(U \cap V) \arrow[r] & \cdots \end{tikzcd}$$

Proof. (a) α is injective since $w \in \Omega^k(U \cup V)$ is determined by $w|_U$ and $w|_V$. Since $\beta \circ \alpha(w) = w|_{U\cap V} - w|_{U\cap V} = 0$, $\operatorname{im}\alpha \subset \ker\beta$. Take $w_1 \in \Omega^*(U)$ and $w_2 \in \Omega^*(V)$ such that $\beta(w_1, w_2) = 0$. Then $w_1|_{U\cap V} = w_2|_{U\cap V}$, therefore just take

$$w = \begin{cases} w_1, & x \in U \\ w_2, & x \in V \end{cases}$$

then $\alpha(w) = (w_1, w_2)$, $\ker\beta \subset \operatorname{im}\alpha$, therefore $\ker\beta = \operatorname{im}\alpha$.

Now show β is surjective. Take $\rho \in \Omega^k(U \cap V)$ and a partition of unity λ_U, λ_V subordinate to the cover $\{U, V\}$. Then $\beta(\lambda_U\rho, -\lambda_V\rho) = (\lambda_U\rho)|_{U\cap V} + (\lambda_V\rho)_{U\cap V} = \rho$.

(b) Denote $d_U^{(k)} \oplus d_V^{(k)} = d_1^{(k)}$, $d_{U\cup V}^{(k)} = d_0^{(k)}$ and $d_{U\cap V}^{(k)} = d_2^{(k)}$. First, claim that $\alpha^{(k)}$ and $\beta^{(k)}$ induces maps on the corresponding cohomology groups. Suppose $w \in \ker d_0^{(k)}$, then $d_1^{(k)} \circ \alpha^{(k)}(w) = \alpha^{(k+1)} \circ d_0^{(k)} = 0$, therefore $\alpha^{(k)}w \in \ker d_1^{(k)}$. Suppose $d_0^{(k)}w \in \operatorname{im} d_0^{(k)}$, then $\alpha^{(k)} \circ d_0^{(k)}(w) = d_1^{(k)} \circ \alpha^{(k+1)}(w) \in \operatorname{im} d_1^{(k)}$, verifying the claim. Then obviously $\alpha_1^{(k)}$, $\beta_1^{(k)}$ and $\alpha_2^{(k)}$, $\beta_2^{(k)}$ induce the same map on the cohomology groups, and $\ker\beta^{(k)} = \operatorname{im}\alpha^{(k)}$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker d_{U\cup V}^{(k)} \arrow[r, "\alpha_1^{(k)}"] \arrow[d] & \ker(d_U^{(k)} \oplus d_V^{(k)}) \arrow[r, "\beta_1^{(k)}"] \arrow[d] & \ker d_{U\cap V}^{(k)} \arrow[d] \arrow[dddll, "\delta"] & \\
0 \arrow[r] & \Omega^k(U\cup V) \arrow[r, "\alpha^{(k)}"] \arrow[d] & \Omega^k(U) \oplus \Omega^k(V) \arrow[r, "\beta^{(k)}"] & \Omega^k(U\cap V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^{k+1}(U\cup V) \arrow[r, "\alpha^{(k+1)}"] \arrow[d] & \Omega^{k+1}(U) \oplus \Omega^{k+1}(V) \arrow[r, "\beta^{(k+1)}"] \arrow[d] & \Omega^{k+1}(U\cap V) \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker} d_{U\cup V}^{(k)} \arrow[r, "\alpha_2^{(k+1)}"] \arrow[d] & \operatorname{coker}(d_U^{(k)} \oplus d_V^{(k)}) \arrow[r, "\beta_2^{(k+1)}"] \arrow[d] & \operatorname{coker} d_{U\cap V}^{(k)} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Next, construct the map δ. Take $w \in \ker d_2^{(k)}$, since β is surjective, there exists $(w_U, w_V) \in \Omega^k(U) \oplus \Omega^k(V)$ such that $\beta(w_U, w_V) = w$. Then $d_1^{(k)}(w_U, w_V) \in \ker\beta^{(k+1)} = \operatorname{im}\alpha^{(k)}$, since $\alpha^{(k)}$ is injective, there exists $\rho \in \Omega^{k+1}(U \cup V)$ such that $\alpha^{k+1}(\rho) = d_1^{(k)}(w_U, w_V)$. Define $\delta([w]) = [\rho]$. Since $\alpha^{(k+2)} d_0^{(k+1)}\rho = d_1^{(k+1)}\alpha^{(k+1)}\rho = 0$, $\rho \in \ker d_0^{(k+1)}$. Claim that it is well-defined on the cohomology groups. First, suppose there are two preimage of w, say w_1 and w_2. Then $w_1 - w_2 \in \ker\beta^{(k)} = \operatorname{im}\alpha^{(k)}$, and $d_0^{(k)}((\alpha^{(k)})^{-1}(w_1 - w_2)) = (\alpha^{(k+1)})^{-1}(d_1^{(k)}(w_1 - w_2))$, hence the choice of preimage does not matter. Second, suppose $w_1 - w_2 = d_2^{(k-1)} \circ \beta^{(k-1)}(\gamma)$, then $w_1 - w_2 = \beta^{(k)} \circ d_2^{(k)}(\gamma)$. Take $d_2^{(k)}(\gamma)$ as the preimage, then obviously it does not affect the image of δ.

Now verify $\ker\delta = \operatorname{im}\beta^{(k)}$. Suppose $w \in \operatorname{im}\beta^{(k)}$, then there exists $\gamma \in \ker d_1^{(k)}$ such that $\beta^{(k)}\gamma = w$. Just take γ as the preimage, then $\delta(w) = 0$, $w \in \ker\delta$. Suppose $w \in \ker\delta$, suppose its image ρ, then there exists γ such that $\rho = d_0^{(k)}\gamma$. So $\alpha^{(k+1)}\rho = d_1^{(k)} \circ \alpha^{(k)}(\gamma)$, therefore the preimage of w under $\beta^{(k)}$ is $\alpha^{(k)}(\gamma) + s$ for some $s \in \ker d_1^{(k)}$. Then $w = \beta^{(k)}(\alpha^{(k)}(\gamma) + s) = \beta^{(k)}(s) \in \operatorname{im}\beta^{(k)}$.

Finally, check $\ker \alpha^{(k+1)} = \operatorname{im} \delta$. Suppose $[w] \in \ker \alpha^{(k+1)}$, then there exists γ such that $\alpha^{(k+1)} w = d_1^{(k)} \gamma$. Then $\delta \circ \beta^{(k)}(\gamma) = w$. Suppose $[w] \in \operatorname{im} \delta$, then obviously $[\alpha^{(k+1)}(w)] = 0$. $\square$

Example 3.45. (a) Let M, N be manifolds with N being homotopic to a point. Then $M \times N$ is homotopic to M: since N is homotopic to a point, $M \times N$ is homotopic to $M \times \{0\}$. Then $M \times \{0\}$ is diffeomorphic to M via $f : M \to M \times \{0\}$, $m \mapsto (m, 0)$.

(b) Compute the de Rham cohomology for spheres S^n. When $n = 1$, by previous examples $H^0_{dR}(S^1) \cong \mathbb{R}$, $H^1_{dR}(S^1) \cong \mathbb{R}$. For $n = 2$, take $S^2 = U \cup V$ where $U = \{(x, y, z) \in S^2 \mid z > -1/2\}$ and $V = \{(x, y, z) \in S^2 \mid z < 1/2\}$. Notice that U and V are homotopic to points. Then there is the Mayer-Vietoris sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0_{dR}(S^2) \arrow[r, "\alpha^{(0)}"] & H^0_{dR}(pt) \oplus H^0_{dR}(pt) \arrow[r, "\beta^{(0)}"] & H^0_{dR}(S^1) \arrow[dll, "\delta^{(0)}"] \\
& H^1_{dR}(S^2) \arrow[r, "\alpha^{(1)}"] & H^1_{dR}(pt) \oplus H^1_{dR}(pt) \arrow[r, "\beta^{(1)}"] & H^1_{dR}(S^1) \arrow[dll, "\delta^{(1)}"] \\
& H^2_{dR}(S^2) \arrow[rr] & & 0
\end{tikzcd}$$

Since $\ker \beta^{(0)} = \operatorname{im} \alpha^{(0)}$ is one dimensional, $\operatorname{im} \beta^{(0)}$ is also one-dimensional and therefore surjective. Hence $\delta^{(0)} = 0$, $\alpha^{(1)}$ is injective, thus $H^1_{dR}(S^2) = 0$. Since $\beta^1 = 0$, $\delta^{(1)}$ is bijective. Thus $H^2_{dR}(S^2) = \mathbb{R}$. For S^n, since S^n is connected, $H^0_{dR}(S^n) \cong \mathbb{R}$. Take $U = S^n \backslash \{(1, 0, \cdots, 0)\}$ and $V = S^n \backslash \{(-1, 0, \cdots, 0)\}$. U and V are homotopic to a point, and $U \cap V$ is diffeomorphic to $S^{n-1} \times \mathbb{R}$. Then there is the following exact sequence for $k \geq 1$:

$$\cdots \longrightarrow 0 \longrightarrow H^k_{dR}(S^{n-1}) \longrightarrow H^{k+1}_{dR}(S^n) \longrightarrow 0 \longrightarrow \cdots$$

therefore $H^k_{dR}(S^{n-1}) \cong H^{k+1}_{dR}(S^n)$. For $k = 0$,

$$0 \longrightarrow \mathbb{R} \xrightarrow{f} \mathbb{R} \oplus \mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{h} H^1_{dR}(S^n) \longrightarrow 0 \longrightarrow \cdots$$

obviously $\operatorname{im} f = \mathbb{R} = \ker g$, therefore g is surjective, thus $h = 0$. Since h is surjective, $H^1_{dR}(S^n) = 0$. Therefore

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R}, & k = 0, n \\ 0, & \text{otherwise} \end{cases}$$

(c) Since Mobius strip is homotopic to S^1, its homology groups are the same as S^1.

(d) Compute the de Rham cohomology of $M = S^1 \times S^1$. Take U and V to be the upper and lower part of the torus, then U and V are both homotopic to S^1, and $U \cap V$ is homotopic to $S^1 \sqcup S^1$. Therefore there is the sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0_{dR}(M) \arrow[r] & H^0_{dR}(U \sqcup V) \arrow[r, "\beta^0"] & H^0_{dR}(U \cap V) \arrow[dll, "\delta^0"] \\
& H^1_{dR}(M) \arrow[r, "\alpha^1"] & H^1_{dR}(U \sqcup V) \arrow[r, "\beta^1"] & H^1_{dR}(U \cap V) \arrow[dll, "\delta^1"] \\
& H^2_{dR}(M) \arrow[r] & 0 &
\end{tikzcd}$$

suppose the inclusion maps $\iota_U : U \cap V \to U$ and $\iota_V : U \cap V \to V$. Then since the restriction of constant functions on U gives the same constant on both components of $U \cap V$ and similarly for V, for constant functions $(a,b) \in H^0_{dR}(S^1) \oplus H^0_{dR}(S^1)$, $\beta^0(a,b) = \iota_V^* b - \iota_U^* a = (b-a, b-a)$. Denote the two components of $U \cap V$ to be A and B respectively. Obviously A and B both are retracts of U and V. Take an embedding of S^1 on A such that $d\theta$ induces a generator on $H^1_{dR}(A)$, then it also induces a generator on $H^1_{dR}(U)$. Therefore in the following diagram the map from $H^1_{dR}(U)$ to $H^1_{dR}(A)$ and $H^1_{dR}(B)$ is just the identity. Therefore β^0 maps $(bd\theta, ad\theta)$ to $((b-a)d\theta, (b-a)d\theta)$, the image is one dimensional.

$$\begin{tikzcd}
H^1_{dR}(U) \arrow[d] & H^1_{dR}(A) \\
H^1_{dR}(U \sqcup V) \arrow[r, "\beta^1"] & H^1_{dR}(U \cap V) \arrow[u] \arrow[d] \\
H^1_{dR}(V) \arrow[u] & H^1_{dR}(B)
\end{tikzcd}$$

Therefore $H^2_{dR}(M) = H^1_{dR}(U \cap V)/\ker \delta^1 = H^1_{dR}(U \cap V)/\operatorname{im} \beta^1 \cong \mathbb{R}$, $H^1_{dR}(M) = \ker \alpha^1 \oplus \operatorname{im} \alpha^1 = \operatorname{im} \delta^0 \oplus \ker \beta^1 \cong (H^1_{dR}(U \cap V)/\operatorname{im} \beta^0) \oplus \mathbb{R} \cong \mathbb{R} \oplus \mathbb{R}$, and since M is connected, $H^0_{dR}(M) = \mathbb{R}$.

(e) Compute the de Rham cohomology of $M = K$, the Klein bottle. Divide it into two cylinders U, V, then their intersection $U \cap V$ is also a cylinder. Write the Mayer-Vietoris sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0_{dR}(M) \arrow[r] & H^0_{dR}(U \sqcup V) \arrow[r, "\beta^0"] & H^0_{dR}(U \cap V) \arrow[dll, "\delta^0"'] \\
& H^1_{dR}(M) \arrow[r, "\alpha^1"] & H^1_{dR}(U \sqcup V) \arrow[r, "\beta^1"] & H^1_{dR}(U \cap V) \arrow[dll, "\delta^1"'] \\
& H^2_{dR}(M) \arrow[r] & 0 &
\end{tikzcd}$$

Obviously $\operatorname{im} \beta^0 = \mathbb{R}$. To determine β^1, again one looks at the generator given by $d\theta$ from S^1 on $U \cap V$. Since U and V are opposite cylinders, the mapping is just $d\theta \mapsto (d\theta - (-d\theta)) = 2d\theta$, therefore $\operatorname{im} \beta^1 = \mathbb{R}$. Hence $H^2_{dR}(M) = H^1_{dR}(U \cap V)/\ker \delta^1 = H^1_{dR}(U \cap V)/\operatorname{im} \beta^1 \cong 0$, $H^1_{dR}(M) = \ker \alpha^1 \oplus \operatorname{im} \alpha^1 = \operatorname{im} \delta^0 \oplus \ker \beta^1 = 0 \oplus \mathbb{R} = \mathbb{R}$. Since the Klein bottle is connected, $H^0_{dR}(M) = \mathbb{R}$.

(f) Compute the de Rham cohomology of $M = \mathbb{RP}^n$. First introduce a map $A : S^n \to S^n, x \mapsto -x$. For $[w] \in H^n_{dR}(S^n)$, $\int_{S^n} w$ is an isomorphism to $\mathbb{R}$ (since by Stokes' theorem exact forms will be sent to zero). Define w by $w_p(v_1, \cdots, v_{n-1}) = \det(p, v_1, \cdots, v_{n-1})$. Then w is a generator. Since A sends w to $(-1)^{n+1}w$, A^* is just a multiplication map by $(-1)^{n+1}$. Since $(A^*)^2 = \mathrm{id}$, the complexes of S^n, $\Omega^k(S^n)$, can be decomposed into the direct sum of eigen spaces of A^*, i.e. $\Omega^k(S^n) = \Omega^k(S^n)_+ \oplus \Omega^k(S^n)_-$ via $v = (1/2)(v + Pv) + (1/2)(v - Pv)$. Since d respects the decomposition, one can define $H^k_{dR}(S^n)_+$ and $H^k_{dR}(S^n)_-$. Obviously $H^n_{dR}(S^n) \cong H^n_{dR}(S^n)_+$ when n is odd, and $H^n_{dR}(S^n) \cong H^n_{dR}(S^n)_-$ when n is even. Suppose the projection map $\pi : S^n \to \mathbb{RP}^n$, since $\pi A = \pi$, $\pi^* = A^* \pi^*$. Thus $\pi^* : \Omega^k(\mathbb{RP}^n) \to \Omega^k(S^n)_+$. Claim that π^* is an isomorphism. If the claim is true, then

$$H^k_{dR}(\mathbb{RP}^n) \cong H^k_{dR}(S^n)_+ = \begin{cases} \mathbb{R}, & \text{if } k = 0 \text{ and } k = n \text{ when } n \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

Now verify the claim. Suppose $a \in \Omega^k(S^n)$ and $w \in \Omega^k(\mathbb{RP}^n)$ such that $\pi^* w = a$, then

$$(\pi^* w)_p(v_1, \cdots) = w_{\pi(p)}((\pi_*)_p v_1, \cdots) = a_p(v_1, \cdots)$$

but for every $\pi(p)$ and $(\pi_*)_p v_1$, there are only two choices, one is p, v_1 and the other is $A(p), A_*(v_1)$. Hence w is uniquely determined iff $a_p(v_1, \cdots) = a_{A(p)}(A_* v_1, \cdots) = A^* a_p(v_1, \cdots)$, i.e. $a \in \Omega^k(S^n)_+$.

Theorem 3.46. *Suppose $n \geq 0$ an even integer. Then there is no nowhere vanishing smooth vector field on S^n.*

Proof. Suppose there is such vector field, denoted by $x \mapsto v_x$. Then

$$H(x,t) : S^n \times [0,1] \to S^n, \quad (x,t) \mapsto x\cos(\pi t) + \frac{v_x}{|v_x|}\sin(\pi t)$$

is a smooth homotopy between id and $A : x \mapsto (-x)$. But A is of degree -1, contradiction. $\square$

4. Integration

Definition 4.1. Let V be a finite dimensional real vector space. Two (ordered) basis $\underline{v} = \{v_1, \cdots, v_n\}$ and $\underline{w} = \{w_1, \cdots, w_n\}$ have the same *orientation* if the base change matrix from V to W has positive determinant. This is an equivalent relation on the set of ordered basis, denote their quotient with $\mathrm{or}(V)$.

Example 4.2. (a) $V = \mathbb{R}v$, then $\mathrm{or}(V) = \{[v], [-v]\}$.
(b) $V = \mathbb{R}v_1 \oplus \mathbb{R}v_2$, then $\mathrm{or}(V) = \{[v_1, v_2], [v_2, v_1]\}$.

Definition 4.3. A map $O : M \to \bigsqcup_{p\in M} \mathrm{or}(T_pM)$ is called an *orientation* on M if $\forall p \in M$, $\exists(\varphi, U)$ around p such that $\forall q \in U$, there is

$$O(q) = \left[\left(\frac{\partial}{\partial x_1}(q), \cdots, \frac{\partial}{\partial x_n}(q)\right)\right]$$

Denote $\mathrm{or}(M)$ the set of orientations on M. M is called *orientable* if $\mathrm{or}(M) \neq \varnothing$.

Proposition 4.4. *If M is connected, then the number of orientations on M can only be zero or 2.*

Proof. Suppose O, O' two orientations on M, put $U_= = \{p \in M \mid O(p) = O'(p)\}$ and $U_{\neq} = \{p \in M \mid O(p) \neq O'(p)\}$. By definition 4.3, they are open sets and $M = U_= \cup U_{\neq}$. Therefore $M = U_=$ or $M = U_{\neq}$. Take a point $p_0 \in M$, then the map $\mathrm{or}(M) \to \mathrm{or}(T_{p_0}M)$, $O \mapsto O_{p_0}$ is injective, therefore $|\mathrm{or}(M)| \leq 2$.

Now assume $|\mathrm{or}(M)| = 1$, take the only orientation O. For O, suppose for every point p, the chart around p is $\varphi = (x_1, \cdots, x_n)$. Now take $\varphi' = (x_2, x_1, \cdots, x_n)$ and define O' via

$$p \mapsto \left[\left(\frac{\partial}{\partial x_2}\right)(p), \left(\frac{\partial}{\partial x_1}\right)(p), \cdots\right]$$

then obviously O' is another orientation of M, contradiction. □

Proposition 4.5. *Let M^m be a manifold. Then the following statements are equivalent:*
(a) M is orientable;
(b) M has an atlas where for all the transition maps, their jocabian have positive determinant;
(c) There exists $w \in \Omega^m(M)$ such that for all $p \in M$, $w_p \neq 0$.

Proof. Suppose M is orientable, then just take the atlas using local charts (φ, U) in definition 4.3. Suppose a point p in the intersection of two such charts $U \cap V$, then $[\frac{\partial}{\partial x_1}(p), \cdots, \frac{\partial}{\partial x_n}(p)]$ and $[\frac{\partial}{\partial y_1}(p), \cdots, \frac{\partial}{\partial y_n}(p)]$ must be in the same equivalence class, therefore their base change map (which is the jocabian) must has positive determinant. Conversely suppose there is already such an atlas. Then just take an orientation via $p \mapsto [\frac{\partial}{\partial x_1}(p), \cdots, \frac{\partial}{\partial x_n}(p)]$, it is obviously an orientation and well-defined on the intersections. Hence (a) and (b) are equivalent.

(b)⇒(c). Suppose such an atlas denoted by (φ_i, U_i). Take a partition of unity λ_i subordinate to this atlas. Suppose $\varphi_i = (x_i^1, \cdots, x_i^n)$, put $w = \sum_I \lambda_i dx_i^1 \wedge \cdots \wedge dx_i^n$. For all $p \in M$, suppose local chart φ_j and a tangent vector $v_p = [\frac{\partial}{\partial x_j^1}, \cdots, \frac{\partial}{\partial x_j^n}]$, since the determinants are all positive, $dx_i^1 \wedge \cdots \wedge dx_i^n(v_p) > 0$. Therefore $w_p \neq 0$.

(c)⇒(a). Take $w \in \Omega^m(M)$ nowhere vanishing. Define $O : M \to \bigsqcup_{p\in M} \mathrm{or}(T_pM)$ via $O(p) = \{[\underline{v}] \mid v_1, \cdots, v_m \in T_pM, w(v_1, \cdots, v_m) > 0\}$. Now verify O is indeed an orientation. Take $p \in M$ and a chart (φ, U) around p (U is connected), then $w|_U = a dx_1 \wedge \cdots \wedge dx_n$ with $a \in C^\infty(U, \mathbb{R})$ nowhere vanishing. Then $a(U) \subset (0, \infty)$ or $a(U) \subset (-\infty, 0)$. For the first case, $O(q) = [\frac{\partial}{\partial x_1}(q), \cdots, \frac{\partial}{\partial x_n}(q)]$ for all $q \in U$, therefore (φ, U) satisfies definition 4.3. For the second case, just take chart $(\varphi' = (-x_1, \cdots, x_m), U)$. □

Proposition 4.6. *Let M^m ($m \geq 2$) be a manifold and $\partial M \neq \varnothing$. Suppose $\mathrm{or}(M) \neq \varnothing$, then $\mathrm{or}(\partial M) \neq \varnothing$.*

Proof. Take $O \in \text{or}(M)$, for $p \in \partial M$ and a chart $(\varphi = (x_1, \cdots, x_m), U)$ preserving O and maps U to the lower half space, $U \cap \partial M$ to an open subset of $x_1 = 0$. Since $((x_2, \cdots, x_m), U)$ is a chart for ∂M, take $\partial O(p) = [\frac{\partial}{\partial x_2}(p), \cdots, \frac{\partial}{\partial x_n}(p)]$, then $\partial O \in \text{or}(\partial M)$. $\square$

Example 4.7. (a) $\mathbb{R}^n$ is orientable because $w = dx_1 \wedge \cdots \wedge dx_n$ is nowhere vanishing for all $p \in \mathbb{R}^n$ since $w_p(\frac{\partial}{\partial x_1}(p), \cdots, \frac{\partial}{\partial x_n}(p)) = 1$.

(b) S^n is orientable, because S^n is a boundary of an orientable manifold $S^n = \partial\overline{B_1}(\underline{0})^{(3)}$.

(c) The Mobius strip is not orientable.

Proof. Assume the Mobius strip, denoted by M, is orientable. Then there exists $w \in \Omega^2(M)$ such that w is nowhere vanishing. Now take $E_1(\theta), E_2(\theta) : \mathbb{R} \to TM$ from example 1.34, then $(E_1(\theta), E_2(\theta))$ is an ordered basis of $T_{p(0,\theta)}$ for all $\theta \in \mathbb{R}$. Put $f(\theta) = w_{p(0,\theta)}(E_1(\theta), E_2(\theta))$, then $f : \mathbb{R} \to \mathbb{R}$ is nowhere vanishing. But $f(0) = -f(2\pi)$, contradiction. $\square$

Remark 4.8. One also need the notion of orientability of a zero dimensional manifold. For a zero dimensional manifold M, which is just a countable set of points, an orientation on M is a map $O : M \to \{\pm 1\}$. Then proposition 4.6 still holds: for a one dimensional manifold M and $p \in \partial M$, take a chart such that $\sigma(p) = [\frac{\partial}{\partial x}(p)]$. Take $\partial O(p) \in \{\pm 1\}$ such that $\partial O(p)\frac{\partial}{\partial x}$ is pointing outwards.

Definition 4.9. Suppose M^m a manifold and O an orientation on M. Denote $C_c(M, \mathbb{R})$ and $\Omega_c^k(M)$ to be the collection of continuous functions from $M \to N$ and the differential forms on M with compact support respectively. An atlas $(\varphi_i, U_i)_I$ is called to have orientation O if every chart (φ_i, U_i) has orientation $O|_{U_i}$, i.e. $\forall p \in U_i$, $O(p) = [\frac{\partial}{\partial y_1}(q), \cdots, \frac{\partial}{\partial y_m}(q)]$. For such chart (φ, V), define $I_{(\varphi,V)} : \Omega_c^m(V) \to \mathbb{R}$ via $I_{(\varphi,V)}(ady_1 \wedge \cdots dy_n) = \int_{\varphi(V)} ady_1 \cdots dy_n$. If $m \geq 1$, define $I_{(M,O)} : \Omega_c^m(M) \to \mathbb{R}$ as follows: let $(U_i, \varphi_i)_{i \in I}$ be a locally finite atlas with orientation O and $(\lambda_i)_{i\in I}$ be a partition of unity subordinate to $(U_i)_{i \in I}$. Then $I_{(M,O)}(w) = \sum_{i \in I} I_{(\varphi_i, U_i)}(\lambda_i w)$. If $m = 0$, then $I_{M,O}(a) = \sum_{p \in M} O(p)a(p)$ for $a \in C_c(M, \mathbb{R})$. $I_{(M,O)}(w)$ is called the *integration* of w over M and is denoted by $\int_{(M,O)} w$.

Remark 4.10. $\operatorname{supp} \lambda_i \cap \operatorname{supp} w \neq \varnothing$ only for finitely many $i \in I$, say $i \in I_0$ (this is because $(\operatorname{supp} \lambda_i)_I$ is locally finite, and $\operatorname{supp} w$ is compact). Therefore the summation is finite.

Proposition 4.11. *The integration $I_{(M,O)}$ is well-defined, i.e. it does not rely on the choice of oriented atlas and partition of unity.*

Proof. First suppose two charts (φ, U), (ψ, V) with two coordinates $\varphi = (x_1, \cdots, x_m)$ and $\psi = (y_1, \cdots, y_m)$ with the same orientation, i.e. $\det(D(\psi \circ \varphi^{-1})) > 0$. Then on $U \cap V$, there are two charts φ and ψ. Suppose $w = adx_1 \wedge \cdots \wedge dx_m = bdy_1 \wedge \cdots \wedge dy_m$, put $D = D(\psi \circ \varphi^{-1})$, then $dy_1 \wedge \cdots \wedge dy_m = \det(D)dx_1 \wedge \cdots dx_m$. Therefore

$$adx_1 \wedge \cdots \wedge dx_m = b\det(D)dx_1 \wedge \cdots \wedge dx_m$$

therefore $b\det(D) = a$. In calculus it is shown that

$$\int_{\varphi(U\cap V)} adx_1 \cdots dx_m = \int_{\psi(U \cap V)} a \left|\det\left(\frac{\partial(x_1, \cdots, x_m)}{\partial(y_1, \cdots, y_m)}\right)\right| dy_1 \cdots dy_m$$

since they have the same orientation,

$$\det\left(\frac{\partial(x_1, \cdots, x_m)}{\partial(y_1, \cdots, y_m)}\right) > 0$$

therefore

$$\begin{aligned}\int_{\varphi(U\cap V)} a dx_1 \cdots dx_m &= \int_{\psi(U\cap V)} a \det\left(\frac{\partial(x_1,\cdots,x_m)}{\partial(y_1,\cdots,y_m)}\right) dy_1\cdots dy_m \\ &= \int_{\psi(U\cap V)} a(\det(D))^{-1} dy_1\cdots dy_m \\ &= \int_{\psi(U\cap V)} b dy_1\cdots dy_m\end{aligned}$$

Also, on $U\cap V$, there is $\int_{U\cap V}(a+b) = \int_{U\cap V} a + \int_{U\cap V} b$ by definition. Now suppose two atlases $(\varphi_\alpha, U_\alpha)$ and (ψ_β, U_β) with partition of unity λ_α and ρ_β. Suppose a form $w \in \Omega_c^k(M)$, then

$$\sum_\alpha \int_{(\varphi_\alpha,U_\alpha)} \lambda_\alpha w = \sum_\alpha \int_{(\varphi_\alpha,U_\alpha)} \lambda_\alpha(\sum_\beta \rho_\beta) w = \sum_\alpha\sum_\beta \int_{(\varphi_\alpha,U_\alpha)} \lambda_\alpha\rho_\beta w$$

since $\operatorname{supp}(\lambda_\alpha\rho_\beta) \subset U_\alpha \cap V_\beta$,

$$\begin{aligned}\sum_\alpha\sum_\beta \int_{(\varphi_\alpha,U_\alpha)} \lambda_\alpha\rho_\beta w &= \sum_\alpha\sum_\beta \int_{(\varphi_\alpha,U_\alpha\cap V_\beta)} \lambda_\alpha\rho_\beta w \\ &= \sum_\beta\sum_\alpha \int_{(\psi_\beta,V_\beta)} \lambda_\alpha\rho_\beta w \\ &= \sum_\beta \int_{(\psi_\beta,V_\beta)} \rho_\beta w\end{aligned}$$

the proof is done. □

Example 4.12. (a) Suppose $M = \mathbb{Z} \subset \mathbb{R}$, $\dim M = 0$. Take an orientation $O : M \to \{\pm 1\}, z \mapsto (-1)^z$. Then for $f \in C_c(M,\mathbb{R})$, f is only nonzero on finitely many points (since it is compactly supported). Then

$$\int_{(M,O)} f = \sum_{z\in M} (-1)^z f(z)$$

(b) $M = [a,b] \subset \mathbb{R}$, take two charts $(\mathrm{id} = s, U = [a,c))$ and $(\mathrm{id} = t, U = (b,d])$ where $a < b < c < d$. Suppose $w = (adx)|_M$ where $adx \in \Omega_c^1(\mathbb{R})$. Take a partition of unity λ_U, λ_V. Then

$$\begin{aligned}\int_M adx &= \int_{[a,c)} \lambda_U(s)a(s)ds + \int_{(b,d]} \lambda_V(t)a(t)dt \\ &= \int_a^b a(x)dx\end{aligned}$$

if $a = df$, then by the fundamental theorem of calculus,

$$\int_M adx = f(b) - f(a) = \int_{\partial M} f$$

(c) $M = S^2$, take the natural orientation induced by being the boundary. Then

$$\begin{aligned}\int_{S^2} zdx \wedge dy &= \int_{S^2 \cap \{z>0\}} zdx \wedge dy + \int_{S^2 \cap \{z<0\}} zdx \wedge dy \\ &= \int_{B_1(0)^{(2)}} z_+(x,y)dx \wedge dy + (-1)\int_{B_1(0)^{(2)}} z_-(x,y)dx \wedge dy \\ &= 2\int_{B_1(0)^{(2)}} \sqrt{1-x^2-y^2}dxdy = \frac{4\pi}{3}\end{aligned}$$

Take $N = \overline{B_1(0)^{(3)}}$, with the orientation induced by $\mathbb{R}^3$. Then

$$\int_N dx \wedge dy \wedge dz = \int_N dxdydz = \frac{4\pi}{3} = \int_{\partial N} zdx \wedge dy$$

Theorem 4.13 (Stokes' theorem)**.** *Let $M^m (m \geq 1)$ be an oriented manifold and $w \in \Omega_c^{m-1}(M)$. Then*

$$\int_M dw = \int_{\partial M} w|_{\partial M}$$

Proof. Suppose $w \in \Omega_c^{m-1}(M)$. Let $(\lambda_i)_I$ be a partition of unity subordinate to an locally finite atlas (φ_i, U_i) with the same orientation such that $\overline{U_i}$ is compact for all i. Then $\lambda_i w \in \Omega_c^{m-1}(U_i)$, $\partial U_i = U_i \cap \partial M$ and $\operatorname{supp}\lambda_i \cap \operatorname{supp} w \neq \varnothing$ only for finitely many $i \in I$, say $i \in I_0$. Suppose the stokes' theorem is true for each U_i, then

$$\begin{aligned}\int_M dw &= \int_M d\left(\sum_{i \in I_0} \lambda_i w\right) \\ &= \sum_{i \in I_0} \int_M d(\lambda_i w) = \sum_{i \in I_0} \int_{U_i} d(\lambda_i w) \\ &= \sum_{i \in I_0} \int_{\partial U_i} (\lambda_i w)|_{\partial U_i} = \sum_{i \in I_0} \int_{\partial M} (\lambda_i w)|_{\partial U_i} \\ &= \sum_{i \in I_0} \int_{\partial M} (\lambda_i w)|_{\partial M} \quad (\operatorname{supp}(\lambda_i w|_{\partial M}) \subset \partial U_i) \\ &= \int_{\partial M} dw|_{\partial M}\end{aligned}$$

Now show stokes' theorem for a local chart U. Suppose $w = adx^I$, where $|I| = m-1$. The index that does not appear in I is denoted by j.

Case (a): suppose U is open, then $dw = \frac{\partial a_i}{\partial x_j} dx_j \wedge dx^I$. Since $\partial U = \varnothing$, $\int_{\partial U} w = 0$. Also,

$$\int_U dw = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\partial a_i}{\partial x_j} dx_1 \cdots dx_m (-1)^{j+1}$$

since a is compactly supported, the above integral is just zero by the fundamental theorem of calculus. This concludes case (a).

Case (b): $m \geq 2$, $M = U \subset \{x_1 \leq 0\}$ (the lower half space). Still, take $w = adx^I$ with j the index that does not appear. Then

$$\begin{aligned}\int_U dw &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial a}{\partial x_j} dx_1 \cdots dx_m (-1)^{j+1} \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} a(0, x_2, \cdots, x_m) dx_2 \cdots dx_m \\ &= \int_{\partial U} w|_{\partial U}\end{aligned}$$

this concludes case (b). For the case where $m = 1$, the same arguments works except the orientations are given by $\{\pm 1\}$ on the boundary. With all cases concluded, the proof is done. $\square$

Example 4.14. (a) Fundamental theorem for line integrals, let C be a smooth curve in $\mathbb{R}^3$ parametrized by $r(t) = (x(t), y(t), z(t))$ such that $r'(t) \neq 0$ for all $t \in [a, b]$. Then for $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$,

$$\int_C df = \int_{\partial C} f = f(r(b)) - f(r(a))$$

(b) Green's theorem. Let D be a compact manifold of $\mathbb{R}^2$ of dimension 2, and $P, Q \in C^\infty(\mathbb{R}^2, \mathbb{R})$. Then

$$\int_{\partial D} Pdx + Qdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

5. Compactly Supported de Rham Complex

Definition 5.1. Let M^m be a manifold. The complex

$$0 \longrightarrow \Omega_c^0(M) \xrightarrow{d_c^{(0)}} \Omega_c^1(M) \xrightarrow{d_c^{(1)}} \cdots \longrightarrow \Omega_c^m(M) \xrightarrow{d_c^{(m)}} 0$$

is called the *compactly supported de Rham complex.* For $k \geq 0$, define $H_c^k(M)$ to be the *compactly supported de Rham cohomology.*

Example 5.2. (a) $H_c^0(\mathbb{R}^n) = 0$ for $n > 0$, and $H_c^0(pt) = \mathbb{R}$;

(b) $H_c^1(\mathbb{R}) \cong \mathbb{R}$. First show that $\mathrm{im}\,(d_c^{(0)}) = \{w \in \Omega_c^1(\mathbb{R}) \mid \int_{\mathbb{R}} w = 0\}$. Suppose $dw \in \mathrm{im}\, d_c^{(0)}$, then by Stokes' theorem $\int_{\mathbb{R}} dw = \int_{\partial \mathbb{R}} w = 0$. For the other direction, suppose $w \in \Omega_c^1(\mathbb{R})$ with $\int_{\mathbb{R}} w = 0$. Since $H_{dR}^1(\mathbb{R}) = 0$, there exists $a \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $w = da$. Take $[c,d]$ such that $\mathrm{supp}\, w \subset (c,d)$. Then

$$0 = \int_{\mathbb{R}} w = \int_{[c,d]} w = a(d) - a(c)$$

take $e = a(d)$, then $a(x) = e$ for all $x \in \mathbb{R} \backslash (c,d)$. Take $b(x) = a(x) - e$, then $db = w$ and b has compact support. Therefore the map $H_c^1(\mathbb{R}) \to \mathbb{R}$, $w \mapsto \int_{\mathbb{R}} w$ injective, and it is obviously surjective.

(c) $H_c^1(\mathbb{R}^2) = 0$. Suppose $\alpha \in \Omega_c^1(\mathbb{R}^2)$ with $d\alpha = 0$. Since $H_{dR}^1(\mathbb{R}^2) = 0$, there exists $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ such that $df = \alpha$. Take a curve C from P to Q with $C' \neq 0$ everywhere and $\mathrm{im}\,(c) \cap \mathrm{supp}\, \alpha = \varnothing$, then

$$0 = \int_C \alpha|_C = \int_C df|_C = \int_{\partial C} f|_{\partial C} = f(P) - f(Q)$$

and the rest of the argument follows from (b).

(d) $H_c^2(\mathbb{R}^2) \cong \mathbb{R}$. One only need to show that $w \in \Omega_c^2(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} w = 0$ is in $\mathrm{im}\,(d_c^{(1)})$. Suppose $w = adx \wedge dy$ with $\mathrm{supp}\,(a) \subset (-R,R)^2$. Put $A(x,y) = \int_{-R}^x a(t,y)dt$, then $A = 0$ for $|y| \geq R$ and $x \leq -R$. Suppose λ a bump function supported in $(0,1)$ such that $\int_{\mathbb{R}} \lambda dt = 1$, put $C(t) = \int_0^t \lambda(s)ds$ and $B(y) = A(R,y)$. Since $\int_{\mathbb{R}} B(y)dy = \int_{\mathbb{R}^2} w = 0$, there exists $D \in \Omega_c^0(\mathbb{R})$ such that $dD = Bdy$. Let $w_0 = (A - C(x)B(y))dy - \lambda(x)D(y)dx$. Since $\mathrm{supp}\, \lambda D \subset (-R,R)^2$ and $A(x,y) - C(x)B(y) = 0$ for $x \geq R$, $\mathrm{supp}\, w_0 \subset (-R,R)^2$. Also,

$$dw_0 = adx \wedge dy - \lambda(x)B(y)dx \wedge dy - \lambda(x)B(x)dy \wedge dx = adx \wedge dy$$

Theorem 5.3. *Suppose $n, k \geq 0$, then*

$$H_c^k(\mathbb{R}^n) \cong \begin{cases} \mathbb{R}, & k = n \\ 0, & \textit{otherwise} \end{cases}$$

Proof. Attempt proof by showing that for $n \geq 2$ and $k \geq 1$, $H_c^k(\mathbb{R}^n) \cong H_c^{k+1}(\mathbb{R}^{n+1})$ and $H_c^1(\mathbb{R}^n) = 0$. Firstly, $H_c^1(\mathbb{R}^n) = 0$ follows from the exact same argument in example 5.2 (c). Fix a bump function $\lambda : \mathbb{R} \to [0,1]$ with integral 1, and $\Lambda(x) = \int_{-\infty}^x \lambda(x)dx$. Put

$$l_k : \Omega_c^k(\mathbb{R}^n) \to \Omega_c^{k+1}(\mathbb{R}^{n+1}), \quad l_k(a_I dx^I) = a_I \lambda(x_{n+1}) dx^I \wedge dx_{n+1}$$

then

$$d \circ l_k(a_I dx^I) = \lambda(x_{n+1}) \sum_{j \notin I} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx^I \wedge dx_{n+1} = l_{k+1} \circ d(a_I dx^I)$$

also put

$$j_{k+1} : \Omega_c^{k+1}(\mathbb{R}^{n+1}) \to \Omega_c^k(\mathbb{R}^n), \quad j_{k+1}(dx^I) = 0, \quad j_{k+1}(b_J(x, x_{n+1})dx^J \wedge dx_{n+1}) = \int_{-\infty}^{\infty} b_J(x,s)ds dx^J$$

then obviously it also has $d \circ j_{k+1} = j_{k+2} \circ d$. Therefore they induce maps on the respective cohomology groups. Since

$$j_{k+1} \circ l_k(a_I dx^I) = \left(\int_{-\infty}^{\infty} \lambda(x_{n+1}) dx_{n+1} \right) a_I dx^I = a_I dx^I$$

$j_{k+1} \circ l_k = \mathrm{id}_{H_c^k(\mathbb{R}^n)}$. Now show that $l_k \circ j_{k+1}$ is homotopy equivalent to id. Suppose a homotopy operator

$$H_{k+1} : \Omega_c^{k+1}(\mathbb{R}^{n+1}) \to \Omega_c^k(\mathbb{R}^{n+1})$$

$$H_{k+1}(dx^I) = 0, \quad H_{k+1}(b_J(x, x_{n+1}) dx^J \wedge dx_{n+1}) = \int_{-\infty}^{x_{n+1}} b_J(x,s) ds - \Lambda(x_{n+1}) \int_{-\infty}^{\infty} b_J(x,s) ds dx^J$$

claim that $1 - l_k \circ j_{k+1} = (-1)^k (dH_{k+1} - H_{k+2} d)$. Suppose $w = a_I dx^I$, then

$$l_k \circ j_{k+1}(a_I dx^I) = 0, \quad d \circ H_{k+1} = 0$$

and

$$\begin{aligned} H_{k+2} \circ d(a_I dx^I) &= H_{k+2} \left(\sum_{j \notin I} \frac{\partial a_I}{\partial x_j} dx^j \wedge dx^I + \frac{\partial a_I}{\partial x_{n+1}} dx_{n+1} \wedge dx^I \right) \\ &= H_{k+2} \left((-1)^k \frac{\partial a_I}{\partial x_{n+1}} dx^I \wedge dx_{n+1} \right) = (-1)^k a_I dx^I \end{aligned}$$

therefore the formula is true for forms of form $a_I dx^I$. For the other case, suppose $w = b_J dx^J \wedge dx_{n+1}$, then

$$(dH_{k+1} - H_{k+2} d)(b_J dx^J \wedge dx_{n+1}) = (-1)^k (1 - l_k j_{k+1})(b_J dx^J \wedge dx_{n+1})$$

therefore the two maps induces isomorphisms on the cohomology groups, which concludes the proof. □

Remark 5.4. (a) Let M be a manifold that admits one global chart. Then $H_c^k(M) \cong H_c^{k+1}(M \times \mathbb{R})$ for all $k \geq 0$. This follows from the argument for theorem 5.3, since M admits one global chart. Now compute the compactly supported de Rham cohomology for half spaces: $H_c^k(\mathbb{R}^{n-1} \times [0, \infty)) = 0$ for all k. One only need to consider the case where $n = 1$. Since $[0, \infty)$ is not compact and connected, $H_c^0([0, \infty)) = 0$. Suppose $w = a dx \in \Omega_c^1([0, \infty))$, then $w = db$, and since w is compactly supported, suppose $w = 0$ on (R, ∞). Then

$$\int_{[0,\infty)} w = \int_{[0,R]} w = b(R) - b(0)$$

therefore $w = d(b - b(0)) \in \operatorname{im} d$.

(b) From now on only consider manifolds without boundary.

Definition 5.5. Let M be a manifold and $k \geq 0$. A differential k-form $w \in \Omega_c^k(M)$ is called a *bump form* if there exists (φ, U) a chart such that $\operatorname{supp}(w)$ is compact and is a subset of U, with $w = \lambda dx^I$ for some bump function λ.

Theorem 5.6. *Suppose M a n-dimensional connected manifold. Then*

$$H_c^n(M) \cong \begin{cases} \mathbb{R}, & M \text{ is orientable} \\ 0, & \text{otherwise} \end{cases}$$

Proof. Suppose $[w] \in H_c^n(M)$, and an atlas (φ_i, U_i) whose charts are just open balls. Take a partition of unity λ_i subordinate to this atlas. Since $\operatorname{supp} w$ is compact, only for finitely many i, say $\{1, \cdots, s\}$, $w_i = \lambda_i w$

is non-zero. Therefore $w = \sum_{i \in I_0} \lambda_i w$. Since $H_c^n(U_i) \cong \mathbb{R}$ by integration, $[w_i] = c_i[u_i]$ where u_i is some bump form. By lemma 5.7,

$$[w] = \sum_{i=1}^{n} c_i[u_i] = c[u_1]$$

therefore $\dim H_c^n(M) \leq 1$. If M is orientable, by Stokes' theorem, the integration is a well-defined linear map to $\mathbb{R}$. Hence $\dim H_c^n(M) = 1$.

If M is non-orientable, then by lemma 5.8 one may take such charts $(\varphi_1, U_1), \cdots, (\varphi_n, U_n)$. Take $[u_i]$ to be the bump form on U_i with

$$\int_{U_i} u_i = 1$$

By previous arguments $[u_1]$ is a generator of $H_c^n(M)$. By the proof of lemma 5.7, one has

$$[u_1] = [u_2] = \cdots = [u_n] = -[u_1]$$

therefore $[u_1] = 0$, $H_c^n(M) = 0$. □

Lemma 5.7. *Suppose M a n-dimensional connected manifold and w_1, w_2 be two non-zero top bump forms. Then there exists $c \in \mathbb{R}$ such that $[cw_1] = [w_2]$ in $H_c^1(\mathbb{R})$.*

Proof. Denote $BF(M)$ to be the collection of bump forms on M, and put $BF_p(M) = \{w \in BF(M) \mid p \in \operatorname{int} \operatorname{supp} w\}$. Suppose $a, b \in BF_p(M)$, and they can be written as a bump form locally on A, B with $A, B \cong \mathbb{R}^n$ respectively. Take an open ball U around p such that $U \subset A \cap B$, and the chart on U has the same orientation with A. Take a bump form on U, say u, with integration 1. Let

$$\alpha = \int_A a, \quad \beta = \int_B b$$

since $H_c^n(\mathbb{R}^n) \cong \mathbb{R}$ via integration, $[\alpha u] = [a]$ in $H_c^n(A)$ and $[\pm\beta u] = [b]$ in $H_c^n(B)$ where the sign is determined by the orientation of B respective to A. For convenience replace $\pm\beta$ with β. Then

$$\alpha u - a = ds, \quad \beta u - b = dt \quad (*)$$

where $s \in \Omega_c^{n-1}(A)$, $t \in \Omega_c^{n-1}(B)$. Since s, t can be easily extended to $\Omega_c^{n-1}(M)$ by bump functions, $(*)$ is also true in $\Omega_c^n(M)$. Therefore in $H_c^n(M)$, $[a] = \alpha[u]$, $[b] = \beta[u]$, hence $[a] = (\alpha/\beta)[b]$.

Put $X = \{q \in M \mid \exists w \in BF(M)_q$ such that $[w] = c[u]$ for some $c \in \mathbb{R}\}$. X is obviously open, and $M \backslash X$ is also open, since otherwise one may take $r \in M \backslash X$ such that there is a sequence of point r_n in X such that $r = \lim_{n \to \infty} r_n$. Take k large enough such that r_k and r can be contained in the a small open ball. By the argument in the last paragraph it is easy to construct a bump form whose support contains r and cohomologous to u, hence $r \in X$, contradiction.

Since M is connected, $X = M$. This closes the proof. □

Lemma 5.8. *Suppose M a non-orientable connected manifold. Then for all $p \in M$, there exists a finite collection of charts $(\varphi_1, U_1), \cdots, (\varphi_n, U_n)$ with $p \in U_1 = U_n$ such that $U_i \cap U_{i+1} \neq \varnothing$, $\det(D(\varphi_{i+1} \circ \varphi_i^{-1})) > 0$ for $1 \leq i \leq n-1$, and $\det(D(\varphi_1 \circ \varphi_n^{-1})) < 0$.*

Proof. Assume on contrary there does not exist such charts. Take $p \in M$, fix a chart U_1 around p. Then for all $q \in M$, there is a path $c : [0,1] \to M$ with $c(0) = p$ and $c(1) = q$. Note that c can always be taken as injective (if c is not injective at $c(a)$, put $t_0 = \inf\{t \in [0,1] \mid c(t) = c(a)\}$ and $t_1 = \sup\{t \in [0,1] \mid c(t) = c(a)\}$, then define a new curve c_1 where $c_1(t) = c(t)$ for $t \in [0,1] \backslash [t_0, t_1]$ and $c_1(t) = c(a)$ for $t \in [t_0, t_1]$). Since c is compact, it can be covered by finitely many small open balls $U_c^1, \cdots, U_c^r$ such that only the neighbouring open sets have non-empty connected intersections. Adjust the charts on $U_c^1, \cdots, U_c^r$ so that they all admit the same orientation. Then for every $q \in M$, one has a chart $U_c^r = U_c(q)$. Claim this gives an oriented atlas on M, i.e. for all $a, b \in M$ with $U_{c_1}(a) \cap U_{c_2}(b) \neq \varnothing$, they have the same orientation. If they do not have the same orientation, then one may shrink $U_{c_1}(a)$ and $U_{c_2}(b)$ so that their intersection is connected and the jacobian is negative. Then $U^1_{c_1}, \cdots, U^r_{c_1}, U^s_{c_2}, \cdots, U^1_{c_2}$ will be a collection of charts in the lemma, contradiction. $\square$

Theorem 5.9 (Poincaré Duality). *Let M^m be an oriented manifold without boundary such that M admits a finite good cover. Then the pairing*

$$\int_M : H^k_c(M) \times H^{m-k}_{dR}(M) \to \mathbb{R}, \quad ([a],[b]) \mapsto \int_M a \wedge b$$

is non-degenerate, i.e. $H^k_c(M) \cong H^{m-k}_{dR}(M)$.

Proof. First show the well-definedness of the pairing. Suppose $[w_1] = [w_2] \in H^k_c(M)$, then $w_1 - w_2 = da$ for $a \in \Omega^{k-1}_c(M)$. For $\rho \in \Omega^{m-k}(M)$ with $d\rho = 0$, one has $da \wedge \rho = d(a \wedge \rho) - (-1)^{k-1} a \wedge d\rho = d(a \wedge \rho)$. By Stokes' theorem, $\int_M da \wedge \rho = 0$. The same argument goes for the second entry of the pairing.

Suppose finite good cover $U_1, \cdots, U_s$ for M, denote $V_k = \bigcup_{i=1}^k U_i$, attempt proof by induction on k. By lemma 5.13, it is true for $k = 1$. Now assume it is true for V_{k-1}, then there is the following diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\delta^*"] & (H^s_c(V_k))^* \arrow[r, "\iota^*"] & (H^s_c(V_{k-1}))^* \oplus (H^s_c(U_k))^* \arrow[r, "\iota^*"] & (H^s_c(V_{k-1} \cap U_k))^* \arrow[r, "\delta^*"] & \cdots \\
\cdots \arrow[r, "\delta"] & H^{m-s}_{dR}(V_k) \arrow[r, "\iota"] \arrow[u] & H^{m-s}_{dR}(V_{k-1}) \oplus H^{m-s}_{dR}(U_k) \arrow[r, "\iota"] \arrow[u] & H^{m-s}_{dR}(V_{k-1} \cap U_k) \arrow[r, "\delta"] \arrow[u] & \cdots
\end{tikzcd}$$

The top row is reversed via $\pi^*(f) = f \circ \pi$. It is clear that the reversed row is still exact. By lemma 5.12 and lemma 5.13, one just need to show commutativity. Obviously it can be deduced to two cases:

Case (a): For $N \subset M$, consider the diagram

$$\begin{tikzcd}
(H^s_c(M))^* \arrow[r] & (H^s_c(N))^* \\
H^{m-s}_{dR}(M) \arrow[r] \arrow[u] & H^{m-s}_{dR}(N) \arrow[u]
\end{tikzcd}$$

for $w \in H^s_c(N)$, $\iota(w)$ is just the extension by zero, hence $\operatorname{supp} \iota(w) \subset N$. For $\rho \in H^{m-s}_{dR}(M)$, $\iota(\rho) = \rho_N$. Then

$$\int_M \iota(w) \wedge \rho = \int_N w \wedge \rho|_N$$

thus the diagram commute.

Case (b): For M, N two open sets, consider the diagram

$$\begin{tikzcd}
(H^s_c(M \cap N))^* \arrow[r, "\delta^*"] & (H^{s-1}_c(M \cup N))^* \\
H^{m-s}_{dR}(M \cap N) \arrow[r, "\delta"] \arrow[u] & H^{m-s+1}_{dR}(M \cup N) \arrow[u]
\end{tikzcd}$$

take $\rho \in H_{dR}^{m-s}(M)$, put $\widetilde{\rho} = \delta(\rho)$, then $\widetilde{\rho}|_M = d\rho_1$, $\widetilde{\rho}|_N = d\rho_2$ and $\rho = \rho_1|_{M\cap N} - \rho_2|_{M\cap N}$. Also, for $w \in H_c^{s-1}(M \cup N)$, put $\widetilde{w} = \delta(w)$, then $(\iota_{M\cap N}^M)_*\widetilde{w} = dw_1$, $(\iota_{M\cap N}^N)_*\widetilde{w} = dw_2$ such that $w_1 - w_2 = w$. Hence

$$\begin{aligned}\int_{M\cup N} w \wedge \widetilde{\rho} = \int_{M\cup N} (w_1 - w_2) \wedge \widetilde{\rho} &= \int_M w_1 \wedge d\rho_1 - \int_N w_2 \wedge d\rho_2 \\ &= (-1)^s \left(\int_M dw_1 \wedge \rho_1 - \int_N dw_2 \wedge \rho_2\right) \\ &= (-1)^s \int_{M\cap N} \widetilde{w} \wedge (\rho_1|_{M\cap N} - \rho_2|_{M\cap N}) \\ &= (-1)^s \int_{M\cap N} \widetilde{w} \wedge \rho\end{aligned}$$

therefore for each five lemma diagram, let the left two maps be the pairing maps with sign $(-1)^s$, and put other maps to be the normal pairing maps. Then by previous discussions on case (a) and cases (b), this diagram commutes. $\square$

Remark 5.10. Suppose V, W real vector spaces and $b : V \times W \to \mathbb{R}$ a bilinear form. b is said to be *non-degenerate* if $\forall w \in W\backslash\{0\}$, $\exists v \in V$ such that $b(v, w) \neq 0$ and $\forall v \in V\backslash\{0\}$, $\exists w \in W$ such that $b(v, w) \neq 0$. If V, W are finite dimensional, then b is non-degenerate iff $f : V \to W^*, v \mapsto b(v, \bullet)$ is an isomorphism.

Proof. Suppose b is non-degenerate. Take bases of V, W and suppose they are of dimension m, n respectively, then $b(v, w) = v^T Bw$ for some $m\times n$ matrix B. Take $w_1, \cdots, w_m$ the standard basis of W, denote $B_1, \cdots, B_n$ the rows of B, then $b(v, \bullet) = 0$ iff $v^T B_1 = \cdots = v^T B_m = B_1^T v = \cdots B_m^T v = 0$. Since b is non-degenerate, the only solution for the above equation is $v = 0$, therefore $m \geq n$ and B is of rank n. By the same argument on the other entry one finds $m = n$ and B has full rank. Since B is exactly the matrix representation of the map f under the basis of V and the dual basis of W, f is an isomorphism. $\square$

Proposition 5.11. *Let $M = U \cup V$ be a manifold with open subsets U, V. Then:*

(a) the inclusion $\iota_U^M : U \to M$ induces maps $(\iota_U^M)_* : \Omega_c^*(U) \to \Omega_c^*(M)$ and the same goes for other inclusions;

(b) the sequence

$$0 \longrightarrow \Omega_c^*(U \cap V) \xrightarrow{\alpha} \Omega_c^*(U) \oplus \Omega_c^*(V) \xrightarrow{\beta} \Omega_c^*(U \cup V) \longrightarrow 0$$

is exact, with $\alpha = (\iota_{U\cap V}^U)_* \oplus (\iota_{U\cap V}^V)_*$ and $\beta = (\iota_U^M)_* - (\iota_V^M)_*$;

(c) there exists long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H_c^0(U \cap V) \arrow[r, "\alpha"] & H_c^0(U) \oplus H_c^0(V) \arrow[r, "\beta"] & H_c^0(U \cup V) \arrow[dll, "\delta"] & \\ & H_c^1(U \cap V) \arrow[r, "\alpha"] & H_c^1(U) \oplus H_c^1(V) \arrow[r, "\beta"] & H_c^1(U \cup V) \arrow[r] & \cdots \end{tikzcd}$$

Proof. (a) is obvious: since $w \in \Omega_c^*(U)$ is compactly supported, it may be extended to a compactly supported form by zero using partition of unity.

(b) It is clear that α, β are chain maps and α is injective. Take a partition of unity λ_U, λ_V subordinate to U, V. Then for $w \in \Omega_c^*(U \cup V)$, $w = \lambda_U w + \lambda_V w = \beta(\lambda_U w, -\lambda_V w)$, hence β is surjective. Now check $\operatorname{im}\alpha = \ker\beta$.

It is clear that $\beta \circ \alpha = 0$, therefore $\operatorname{im}\alpha \subset \ker\beta$. Suppose $(w_1, w_2) \in \ker\beta$, then $(\iota_U^M)_* w_1 = (\iota_V^M)_* w_2$, hence $w_1|_{U\cap V} = w_1 = w_2$, denote it by w'. Then $\operatorname{supp} w' \subset \operatorname{supp} w_1 \cap \operatorname{supp} w_2$ which is compact in $U \cap V$. Hence $\alpha(w') = (w_1, w_2)$, $\ker\beta \subset \operatorname{im}\alpha$.

The argument of (c) is completely the same as theorem 3.44 (c). □

Lemma 5.12 (Five lemma). *Consider the following commutative diagram of abelian groups with exact rows:*

$$\begin{tikzcd} G_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & G_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & G_3 \arrow[r, "f_3"] \arrow[d, "\alpha_3"] & G_4 \arrow[r, "f_4"] \arrow[d, "\alpha_4"] & G_5 \arrow[d, "\alpha_5"] \\ H_1 \arrow[r, "g_1"] & H_2 \arrow[r, "g_2"] & H_3 \arrow[r, "g_3"] & H_4 \arrow[r, "g_4"] & H_5 \end{tikzcd}$$

if α_1 is surjective, α_5 is injective, and α_2, α_4 are isomorphisms, then α_3 is an isomorphism.

Proof. First show injectivity. Suppose $\alpha_3(s_3) = 0$, then $f_3(s_3) = 0$ since α_4 is an isomorphism. Hence $s_3 \in \ker f_3$, then there exists s_2 such that $s_3 = f_2(s_2)$. Then $g_2 \circ \alpha_2(s_2) = 0$, $\alpha_2(s_2) \in \ker g_2$, there exists h_1 such that $g_1(h_1) = \alpha_2(s_2)$. Since α_1 is surjective, there exists s_1 such that $\alpha_1(s_1) = h_1$. Then $\alpha_2 \circ f_1(s_1) = g_1 \circ \alpha_1(s_1) = \alpha_2(s_2)$, since α_2 is an isomorphism, $f_1(s_1) = s_2$. Then $s_3 = f_2(s_2) = 0$.

Then show surjectivity. Suppose $h_3 \in H_3$. Then $g_4 \circ g_3(h_3) = 0$. Since α_4 is an isomorphism, there exists s_4 such that $\alpha_4(s_4) = g_3(h_3)$. Then $\alpha_5 \circ f_4(s_4) = 0$. Since α_5 is injective, $f_4(s_4) = 0$, there exists s_3 such that $f_3(s_3) = s_4$. Then $g_3 \circ \alpha_3(s_3) = g_3(h_3)$. Therefore $h = \alpha_3(s_3) - h_3 \in \ker g_3$, there exists h_2 such that $h = g_2(h_2)$. Since α_2 is an isomorphism, there exists s_2 such that $h_2 = \alpha_2(s_2)$. Then $\alpha_3(s_3 + f_2(s_2)) = \alpha_3(s_3) + h = h_3$. □

Lemma 5.13. *The pairing map is non-degenerate on $\mathbb{R}^n$.*

Proof. By the first step of theorem 5.9, the pairing map is a well-defined. One only need to consider the pairing map on $H^n_c(\mathbb{R}^n) \times H^0_{dR}(\mathbb{R}^n)$ since other maps are just zero maps between zero spaces. Suppose $w \in H^n_c(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} w \neq 0$, then just take $\rho = 1 \in H^0_{dR}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} w \wedge \rho = \int_{\mathbb{R}^n} w \neq 0$. Now suppose $\rho = c \in H^0_{dR}(\mathbb{R}^n)$ with $c \neq 0$, take $w = \lambda dx_1 \wedge \cdots \wedge dx_n \in H^n_c(\mathbb{R}^n)$ where λ is a bump function with integration 1, then $\int_{\mathbb{R}^n} w \wedge \rho = c \int_{\mathbb{R}^n} w = c \neq 0$. □

Definition 5.14. Let M, N be closed oriented manifolds and $f \in C^\infty(M, N)$. Then f^* induces maps between $H^m_{dR}(N)$ and $H^m_{dR}(M)$. Since they are both homeomorphic to $\mathbb{R}$ via integration, f^* is just multiplication by a real number. It is called the *degree* of f.

Theorem 5.15. *Suppose M, N closed oriented manifolds and f a smooth map between them. Let q be a regular value of f, and let E be the number of elements in $f^{-1}(q)$ counted with a sign according to $\det(D(\psi \circ f \circ \varphi^{-1}))$ (i.e. whether f preserves orientation around that point). Then for all $w \in \Omega^m(N)$,*

$$\int_M f^* w = E \int_N w$$

Proof. First claim that E is finite. Assume not, then $f^{-1}(q)$ contains infinitely many points. Then one may pick a sequence p_i such that they are pairwise different. Since M is compact, one may take a convergent subsequence p_n with $p = \lim_{n\to\infty} p_n$. Since f is continuous, $f(p) = \lim_{n\to\infty} f(p_n) = q$. Since M and N are of the same dimension and q is a regular value, df_p is an isomorphism and therefore f is a diffeomorphism around p, contradiction.

Since M is compact, $f(M)$ is compact and therefore closed. If $f^{-1}(q) = \varnothing$, then $q \in N \backslash f(M)$ and therefore then $f^* w_q = 0$. □

Remark 5.16. (a) Take a bump form w_N on N with integration 1. Then $E = E \int_N w = \int_M f^* w = \deg f$, therefore $\deg f = E$, which is an integer.

(b) By the definition of degree, one has $\deg(f \circ g) = \deg f \cdot \deg g$.

(c) Diffeomorphisms is always of degree ± 1, with the sign determined by whether it is orientation preserving.

(d) The definition of degree can be naturally extended to proper maps.

Theorem 5.17 (Sard theorem)**.** *Let M^m, N^n ($m \geq 0$, $n \geq 1$) be manifolds and $f \in C^\infty(M, N)$. Then the set of critical values of f is a set of measure zero in N.*

Definition 5.18. Let M^m be a manifold such that all of its cohomology groups are finite dimensional. Then

$$\chi(M) = \sum_{k=0}^{m} (-1)^k \dim H^k_{dR}(M)$$

is called the *Euler characteristic* of M.