Definition. Consider two functors $F, G : \mathscr{A} \to \mathscr{B}$. A *natural transformation* is a collection of morphisms $(\alpha_A : F(A) \to G(A))_{A \in \mathscr{A}}$ such that for all morphisms $f : A \to A'$ in $\mathscr{A}$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(A') & \xrightarrow[\alpha_{A'}]{} & G(A') \end{array}$$

Proposition. *Let $\mathscr{A}$ and $\mathscr{B}$ both be categories and $\mathscr{A}$ is small. Then the collection of all natural transformations The functors from $\mathscr{A} \to \mathscr{B}$ and the natural transformations between them constitute a category.*

Proof. Since $\mathscr{A}$ is small, $\{\alpha \mid \forall A \in ob(\mathscr{A}), \alpha \in \text{Hom}(F(A), G(A))\}$ is a set, and the set of all natural transformations is a subset of the power set of this set. Therefore, the set of all natural transformations between two functors constitutes a set. □

Theorem (Yoneda Lemma). *Suppose a functor $F : \mathscr{A} \to \text{Set}$ where $\mathscr{A}$ is an arbitrary category. Then for all $A \in ob(\mathscr{A})$, there exists a bijection*

$$\theta_{F,A} : \text{Nat}(\text{Hom}(A, -), F) \to F(A)$$

Proof. For a natural transformation $\alpha : \text{Hom}(A, -) \Rightarrow F$, define $\theta_{F,A}(\alpha) = \alpha_A(\text{id}_A)$. On the other hand, suppose $a \in F(A)$, define $\tau : F(A) \to \text{Nat}(\text{Hom}(A, -), F)$ by $\tau(a)_B(f) = F(f)(a)$ for $B \in ob(\mathscr{A})$, $f \in \text{Hom}(A, B)$. To see $\tau(a)$ is indeed a natural transformation, one need to show the following diagram commutes:

$$\begin{array}{ccc} \text{Hom}(A, B) & \xrightarrow[\tau(a)_B]{} & F(B) \\ \downarrow{\scriptstyle \text{Hom}(A,g)} & & {\scriptstyle F(g)}\downarrow \\ \text{Hom}(A, C) & \xrightarrow{\tau(a)_C} & F(C) \end{array}$$

which reduces to $F(g \circ f) = F(g) \circ F(f)$ and directly follows from the functoriality of F. It is straightforward to check that τ is the inverse of $\theta_{F,A}$. □

Remark. Suppose a category $\mathscr{A}$ and a morphism $f : A \to B$ in $\mathscr{A}$. Then define a natural transformation

$$\text{Hom}(f, -) : \text{Hom}(A, -) \Rightarrow \text{Hom}(B, -)$$

with $\text{Hom}(f, -)_C(g) = g \circ f$ for $C \in ob(\mathscr{A})$ and $g \in \text{Hom}(B, C)$.

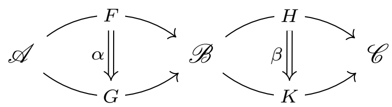
Remark. Consider the functor $N : \mathscr{A} \to \text{Set}$ defined by $N(A) = \text{Nat}(\text{Hom}(A, -), F)$ and $N(f)(\alpha) = \alpha \circ \text{Hom}(f, -)$, there is a natural transformation $\eta : N \Rightarrow F$ defined by $\eta_A = \theta_{F,A}$.

Remark. When $\mathscr{A}$ is small, consider two functors: (1) $M : \text{Fun}(\mathscr{A}, \text{Set}) \to \text{Set}$ defined by $M(F) = \text{Nat}(\text{Hom}(\mathscr{A}, -), F)$ and $M(\gamma)(\alpha) = \gamma \circ \alpha$ for $\gamma : F \Rightarrow G$; (2) $E : \text{Fun}(\mathscr{A}, \text{Set}) \to \text{Set}$ defined by $E(F) = F(A)$ and $E(\gamma) = \gamma_A$. Then $\mu : M \Rightarrow E$ by $\mu_F = \theta_{F,A}$ is a natural transofrmation.

Definition. Given a category $\mathscr{C}$, a functor $F : \mathscr{C} \to \text{Set}$ is said to be a *representable functor* if F is naturally isomorphic to $\text{Hom}(C, -)$ for some $C \in ob(\mathscr{C})$.

Remark. The Yoneda lemma automatically extends to representable functors.

Proposition. *Consider the following situation*



define $(\beta * \alpha)_A = \beta_{GA} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{FA}$, *then* $\beta * \alpha : H \circ F \Rightarrow K \circ G$ *is a natural transformation.*

Proof. For the equality appeared in the definition, it is just the following commutative diagram introduced by β.

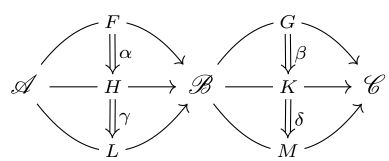
$$\begin{array}{ccc} HF(A) & \xrightarrow{H(\alpha_A)} & HG(A) \\ \downarrow{\scriptstyle \beta_{FA}} & & \downarrow{\scriptstyle \beta_{GA}} \\ KF(A) & \xrightarrow{K(\alpha_A)} & KG(A) \end{array}$$

and the claim that $\beta * \alpha$ is a natural transformation immediately follows from the following commutative diagram, with the two squres being established by the naturality of α and β.

$$\begin{array}{ccccc} HF(A) & \xrightarrow{H(\alpha_A)} & HG(A) & \xrightarrow{\beta_{GA}} & KG(A) \\ \downarrow{\scriptstyle HF(f)} & & \downarrow{\scriptstyle HG(f)} & & \downarrow{\scriptstyle KG(F)} \\ HF(B) & \xrightarrow{H(\alpha_B)} & HG(B) & \xrightarrow{\beta_{GB}} & KG(B) \end{array}$$

□

Remark. Consider the following situation



it is easy to check that $(\delta * \gamma) \circ (\beta * \alpha) = (\delta \circ \beta) * (\gamma \circ \alpha)$.

Remark. When seeing those natural transformation graphs, I cannot hold myself from thinking about homotopies!

Definition. Given a category $\mathscr{A}$, its *opposite category* $\mathscr{A}^{op}$ is obtained by just reversing the morphisms in $\mathscr{A}$. A functor $\mathscr{A}^{op} \to \mathscr{B}$ is also called a *contravariant functor* from $\mathscr{A}$ to $\mathscr{B}$.

Remark. For functors like Hom from $\mathscr{A}$, their *dual* could be obtained by just taking Hom over $\mathscr{A}^{op}$. The same goes for natural transformations over Hom.

Example. Consider Rng the category of commutative rings with units and Top. Note that on the spectrum of the ring one may take the zariski topology. Then by taking the spectrum of the ring, it gives rise to a contravariant functor

$$\mathrm{Sp} : \mathrm{Rng} \to \mathrm{Top}$$

by taking the preimage, since the preimage of a prime ideal is still prime.

I also want to discuss the following questions:

- When we are talking about, say the category of sets, where have we utilized the notion of universes?
- The statement and the proof of the first proposition.